Frontier Topics in Empirical Economics: Week 2
Non-parametric Method

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Non-parametric Method: Introduction

- Common Parametric Models
  Linear Model: $y = X'\beta + e$, $e \sim N(0, \sigma^2)$;
  Probit/Logit Model: $P(y|X) = G(X\beta)$ where G is a nonlinear function
- Explicit Parametric Structure for Distribution
- Common Estimator
  OLS, MLE, Nonlinear LS, Efficient GMM etc.
- Key Properties of the Estimator
  Consistency, BLUE, Asymptotic Efficiency etc.

Non-parametric Method: Introduction

- In linear model, we have to assume that CEF is linear
- Why linear? Simple? Why not $y = \beta x^{3\gamma} \cdot lnx + e$?
- What if linear specification is wrong?
- Everything collapses. No data can save.
- It becomes only a linear approximation
- For example, if true model is Logit, but not linear regression
- Functional form can be wrong

Non-parametric Method: Introduction

- Parametric statistics are based on assumptions about the distribution of population from which the sample was taken
- Non-parametric statistics are NOT based on functional form assumptions
- The data can be collected from a sample that does not follow a specific distribution

Non-parametric Method: Introduction

- Potential Outcome Framework is intrinsically non-parametric
- If we can directly get estimations of $E[y|x=1]$ and $E[y|x=0]$
- We can estimate the ATE/ATT in a more general way without regression
- There are many other statistical modeling methods
- Non-parametric, semi-parametric to estimate CEF directly
- To understand tools beyond linear regression

Non-parametric Method: Introduction

- Let's forget about the model functional form
- Give up the "parametric" model like linear regression
- Do not assume that CEF is linear
- Go back to the original question to estimate $E(y_i|x_i)$ without imposing any functional form assumption

Non-parametric Method: Introduction

- Notation: x_i, y_i denotes random variable; X_i, Y_i denotes realizations; x, y denotes random variables or some value of the random variables
- Realizations are given (sample), they are NOT random in our context
  $\int x \sum_i^n X_i dx = \sum_i^n X_i \int x dx$

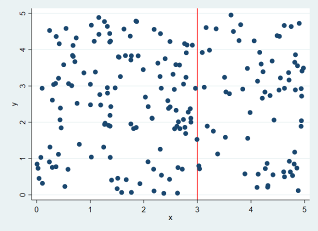
Non-parametric Method: Kernel Regression

- Let's consider the first non-parametric method: Kernel regression
- It is super intuitive and interesting
- Instead of assuming $E(y_i|x_i) = x_i^{\prime}\beta$, we consider this CEF point by point
- That is, estimate $E(y_i|x_i)$ for each possible point of $x_i = x$

Non-parametric Method: Kernel Regression

Step 1: Estimating a cumulative density

- Consider estimating a cumulative density function (CDF)



- What is the CDF at $x = 3$? $\hat{F}(x = 3) = ?$
- Go back to kindergarten!

Non-parametric Method: Kernel Regression

- Just count how many points lie on the left to the red line:

$$\hat{F}(x = 3) = \frac{1}{n} \sum \mathbf{1}(X_i \leq 3)$$

- In general, we have an estimation of $F(x)$ as:

$$F(x) = P(X \leq x) \Rightarrow \hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(X_i \leq x)$$

- The proportion of points (realizations) that are smaller than x

Non-parametric Method: Kernel Regression

Step 2: Estimating a probability density

- Consider estimating a probability density function (PDF)
- PDF represents a marginal increase in CDF at some point (derivative)

$$f(x) = \frac{dF(x)}{dx} = \lim_{h \to 0} \frac{F(x+h) - F(x-h)}{2h}$$

$$\hat{f}(x) = \frac{\hat{F}(x+h) - \hat{F}(x-h)}{2h}$$

- Changes of $F(x)$ in a very small interval (with length $2h$)
- h is called "bandwidth"

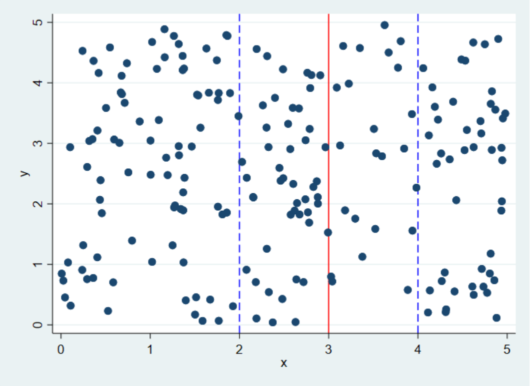
Non-parametric Method: Kernel Regression

- Then we can write the probability density $f(x)$ at some value x as:

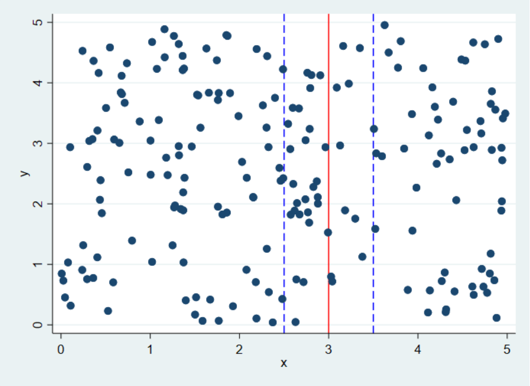
$$\hat{f}(x) = \frac{1}{2h}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(X_i \le x+h) - \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(X_i \le x-h)]$$
$$= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2h}\mathbf{1}(x-h \le X_i \le x+h)$$

- How to interpret this?
- We count the number of obs within a small interval around x, dividing by the length and the total number of obs
- $\sum_{i=1}^{n}\frac{1}{2h}\mathbf{1}(x-h \le X_i \le x+h)$ is the number of obs per unit length
- When n is large, we can choose very small h

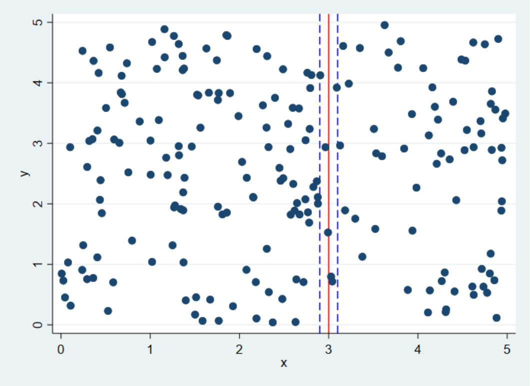
Non-parametric Method: Kernel Regression



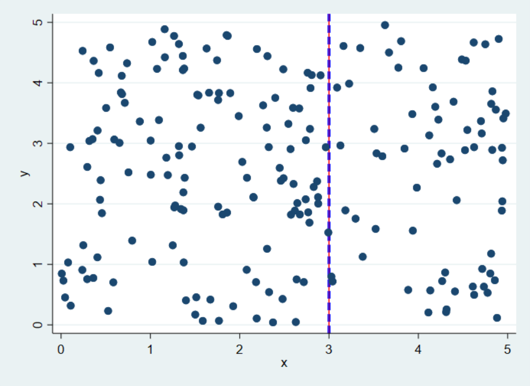
Non-parametric Method: Kernel Regression



Non-parametric Method: Kernel Regression



Non-parametric Method: Kernel Regression



Non-parametric Method: Kernel Regression

- Define $k(v) = \frac{1}{2}\mathbf{1}(|v| \leq 1)$. Then we have:

$$\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}k(\frac{X_i - x}{h})$$

- We call $k(v)$ a uniform kernel function
- This $\hat{f}(x)$ is a kernel estimator of the PDF (uniform kernel)
- Kernel is weight!
- There can be other kinds of kernel functions, when we assign different weights to different observations

Non-parametric Method: Kernel Regression

- A function can be used as a kernel if
  - $k(v)$ is integrated to 1
  - $k(v)$ is symmetric with $k(v) = k(-v)$
- The weights sum to one; The weights are symmetric
- Triangular Kernel: $k(v) = (1 - |v|)\mathbf{1}(|v| \leq 1)$
- Epanechnikov Kernel: $k(v) = \frac{3}{4}(1 - v^2)\mathbf{1}(|v| \leq 1)$
- Gaussian Kernel: $k(v) = \frac{1}{2\pi} e^{\frac{-v^2}{2}}$
- Usually, Epanechnikov Kernel and Triangular Kernel are preferred

Non-parametric Method: Kernel Regression

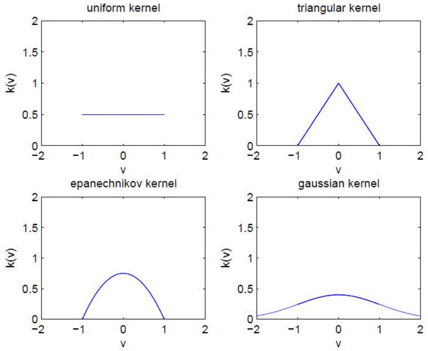


Figure 1: Various Kernels

Non-parametric Method: Kernel Regression

- For multivariate case, let $v = (v_1, v_2, \cdots, v_q)$.
- Define product kernel: $K(v) = k(v_1)k(v_2)\cdots, k(v_q)$.
- The estimator becomes:

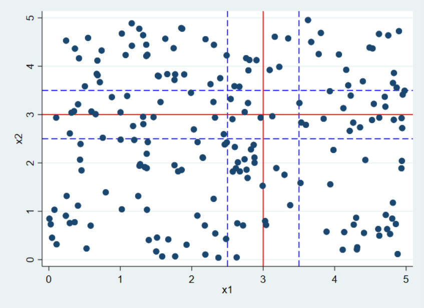
$$\hat{f}(x) = \frac{1}{nh_1 h_2 \cdots h_q} \sum_i K(\frac{X_i - x}{h})$$

- Define $h = (h_1, h_2, \cdots, h_q)$
- $K(\frac{X_i - x}{h})$ is the weighted sum of points within the q-dimension hypercube
- $h_1 h_2 \cdots h_q$ is the volumn of this q-dimension hypercube

Non-parametric Method: Kernel Regression

In two dimension case, we have

- $K(\frac{X_i - x}{h})$ is the weighted sum of points within the rectangular
- $h_1 h_2 \cdots h_q$ is the area of this rectangular



Non-parametric Method: Kernel Regression

Step 3: Estimating a CEF

- Finally, let's see how to estimate a CEF using kernel method
- Not like linear regression, we estimate the CEF point by point
- Assume that we have CEF:

$$Y = g(X) + u$$
$$E[Y|X] = g(X)$$

- u has a conditional variance $Var(u|X) = \sigma^2(x)$

Non-parametric Method: Kernel Regression

Step 3: Estimating a CEF

- Based on the CDF and PDF we've got, we have Nadaraya-Watson Estimator (N-W) for CEF as follows:

$$\hat{g}(x) = \sum_{i=1}^{n} Y_i K_h(X_i - x), \quad \textit{where} \quad K_h(X_i - x) = \frac{K(\frac{X_i - x}{h})}{\sum_{i=1}^{n} K(\frac{X_i - x}{h})}$$

- Intuition: The conditional Expectation of Y given X=x is estimated as a weighted average of observed Y_i closely around x (within the range of bandwidth h).
- Weights are determined by the kernel function

Non-parametric Method: Kernel Regression

Homework:

- 1. Derive NW Estimator from the kernel estimator of CDF and PDF. This can be a little bit hard. You can refer to Notes from Carol (or Hansen's book) for help.
- 2. What is NW Estimator, if we use the uniform kernel?

Non-parametric Method: Kernel Regression

- We have $g(x) = E(Y|X)$ as CEF and $f(x)$ as density for x

Theorem (Asymptotics for N-W Estimator)

Under some regularity conditions, as $n \to \infty, h_s \to 0 (s = 1, \dots, q)$, $nh_1 \dots h_q \to \infty$ *and* $nh_1 \dots h_q \sum_{s=1}^{q} h_s^6 \to 0$, *we have:*

$$\sqrt{nh_1 \dots h_q}(\hat{g}(x) - g(x) - \sum_{s=1}^{q} h_s^2 B_s(x)) \xrightarrow{d} N(0, \frac{\sigma^2(x)}{f(x)}(\int k(v)^2 dv)^q)$$

$$\text{where} \quad B_s(x) = \frac{\int v^2 k(v) dv}{2f(x)}[2\frac{\partial f(x)}{\partial x_s}\frac{\partial g(x)}{\partial x_s} + f(x)\frac{\partial^2 g(x)}{\partial x_s^2}]$$

Non-parametric Method: Kernel Regression

Asymptotic Bias= $\sum_{s=1}^{q} h_s^2 \frac{\int v^2 k(v)dv}{2f(x)}[2\frac{\partial f(x)}{\partial x_s}\frac{\partial g(x)}{\partial x_s} + f(x)\frac{\partial^2 g(x)}{\partial x_s^2}]$

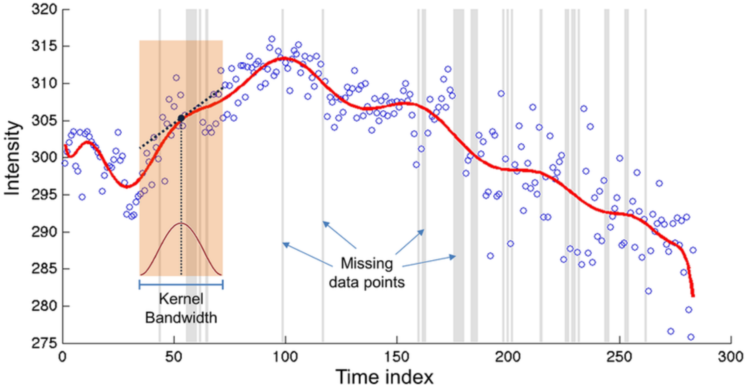
Asymptotic Variance= $\frac{1}{nh_1 \dots h_q}\frac{\sigma^2(x)}{f(x)}(\int k(v)^2 dv)^q$

- (1) $h_s \uparrow \Longrightarrow Bias \uparrow, Variance \downarrow$
  $\therefore$ we have trade-off in choosing kernel bandwidth.
- (2) $q \uparrow \Longrightarrow Variance \uparrow exponentially$
  We call this "Curse of Dimensionality".
- (3) Kernel more concentrated $\Rightarrow Bias \downarrow (\int v^2 k(v)dv), Variance \uparrow (\int k(v)^2 dv))$
- (4) Slope Effect and Curvature Effect on bias: $\frac{\partial f(x)}{\partial x_s}\frac{\partial g(x)}{\partial x_s}, \frac{\partial^2 g(x)}{\partial x_s^2}$
- (5) $f(x) \uparrow \Longrightarrow Bias \downarrow, Variance \downarrow$ (more observations)

Non-parametric Method: Local Polynomial

- Another widely used kernel-based method is local polynomial
- In linear regression, we use a global linear function to fit data
- In local polynomial, we use piece-wise polynomial (linear) function to fit data interval by interval

Non-parametric Method: Local Polynomial



For some $X = x$, we fit $g(x)$ by choosing samples very close to x. Then we fit a polynomial for these observations. (Here, linear)

Non-parametric Method: Local Polynomial

- For $g(x)$, we solve the following optimization problem at each point x:

$$\min_{b_0,b_1,\cdots,b_p} \sum_{i=1}^{n} k(\frac{X_i - x}{h})(Y_i - b_0 - b_1(X_i - x) - b_2(X_i - x)^2 - \cdots - b_p(X_i - x)^p)^2$$

- When $p = 1$, we call it local linear regression
- When $p = 2$, we call it local quadratic regression

Non-parametric Method: Series-based Methods

- Both kernel and local polynomial regressions are Kernel-based methods
- There are three disadvantages of this method:
  - Computational burden is large (point by point estimation)
  - Hard to include information or restriction over functional form
  - Requirement of large sample
- Series-based methods alleviate these problems

Non-parametric Method: Series-based Methods

- As usual, we have a CEF model:

$$Y = g(X) + u$$
$$g(X) = E(Y|X)$$

- We expand the CEF by Taylor Series at zero:

$$g(X) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} X^k$$

Non-parametric Method: Series-based Methods

- This infinite series can be approximated by a K-order global polynomial:

$$g(X) = \sum_{k=0}^{K} \beta_k p_k(X)$$

$$p_0(x) = 1, p_1(x) = x, p_2(x) = x^2, ..., p_K(x) = x^K$$

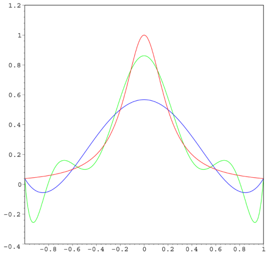
- We can use OLS to estimate this polynomial
- The vector of $\{p_0, p_1, p_2, ..., p_K\}$ is called "basis"
- This is "global" polynomial, in contrast to "local" polynomial

Non-parametric Method: Series-based Methods

- Polynomial is the simplest choice of basis
- In multivariate case (2 variables), it becomes: $\{1, x_1, x_2, x_1x_2, x_1^2, x_2^2, x_1x_2^2, x_1^2x_2, x_1^2x_2^2 ...\}$
- Polynomial series has several problems
- It is very sensitive to outliers
- The biggest problem for polynomial series is Runge's phenomenon

Non-parametric Method: Series-based Methods

- Runge's phenomenon
- Red: original true function; Blue: fifth-order poly; Green: ninth-order poly



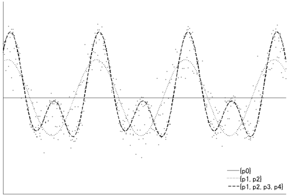
- Since the power polynomials are forced to vary somewhere
- It may be pushed to the boundary
- The boundary part is approximated very poorly

Non-parametric Method: Series-based Methods

- How to choose the optimal order?
- We will discuss this problem in details in the next lecture when considering model selection and machine learning
- But in general, high order polynomial behaves very bad
- Some other basis are better

Non-parametric Method: Series-based Methods

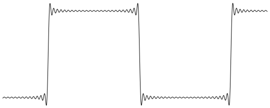
- Fourier basis, derived by Fourier expansion



- Excellent for approximating periodic functions
- Better than poly, but still not good at boundary/jumping point (Gibbs' phenomenon)

Non-parametric Method: Series-based Methods

- Better than poly, but still not good at boundary/jumping point (Gibbs' phenomenon)
- Let's see an approximation of Fourier series to the square wave



Non-parametric Method: Series-based Methods

- There are more basis
- Such as Spline basis and Wavelet basis
- They are complicated, rarely seen in Applied works
- But Carol claims that Spline basis is in general a better choice
- If interested, you can read her notes

Non-parametric Method: Semi-parametric Model

- Non-parametric model is so general that we do not impose any structure
- Totally data driven, no prior information
- Convergence rate is low, variance is high, requirement for data is high
- What if we want to impose some structure, but not the full structure?
- Semi-parametric model

Non-parametric Method: Semi-parametric Model

- Partially linear model
- One of the most popular semi-parametric models

$$Y = X'\beta + g(Z) + u, \quad E(u|X,Z) = 0, Var(u|X,Z) = \sigma^2$$

- X enters in the model linearly, Z non-parametrically

Non-parametric Method: Semi-parametric Model

- Estimation of β is simple, we follow Robinson (1988)
- In the first step, conditional on Z and then take the subtract:

$$E(Y|Z) = E(X^{'}|Z)\beta + g(Z)$$
$$Y - E(Y|Z) = [X - E(X|Z)]^{'}\beta + u$$

- $E(Y|Z)$ and $E(X|Z)$ can be estimated using methods introduced previously
- Then we have estimators for $Y - E(Y|Z)$ and $X - E(X|Z)$
- Then we can estimate β using OLS
- Asymptotics of this estimator is complicated

Non-parametric Method: Semi-parametric Model

- In the second step, we subtract $X'\beta$ from Y:

$$Y - X'\beta = g(Z) + u$$

- g(Z) can be estimated using methods introduced previously

Non-parametric Method: Semi-parametric Model

- Question: How to estimate the variance of $\hat{g}(Z)$?
- Can we use the variance from the non-parametric regression directly?
- No! Because $Y - X'\beta$ is also estimated
- It contains more uncertainty from the first step
- This is a common mistake in empirical work:
  When you have first stage estimation as known parameter in the second stage, watch out for the std err estimation!

Non-parametric Method: Semi-parametric Model

- Similarly, how to conduct inference for first step β?
- It is a combination of non-parametric and regression estimations
- No closed-form variance equation is available
- Not possible to directly calculate the standard error

Non-parametric Method: Semi-parametric Model

In these two cases, we need bootstrap for inference

Non-parametric Method: Bootstrap

- Bootstrap is a non-parametric method for inference
- It is used when there is no closed-form standard errors
- Instead of deriving the closed-form equation of variance
- We use simulation to estimate it
- Random sampling with replacement

Non-parametric Method: Bootstrap

- Step 1: Given full sample with size n, draw R new samples of size n, with replacement. Index each new sample by r
- Step 2: Calculate the simulated variance of $\hat{g}(x)$ by: $\hat{V}(x) = \frac{1}{R-1}\sum_{r=1}^{R}[\hat{g}_r(x) - \hat{g}(x)]^2$
- Step 3: Use $\hat{V}(x)$ to calculate confidence intervals and implement statistical tests
- We call this bootstrapped variance

Non-parametric Method: Bootstrap

- But using bootstrapped variance to construct confidence interval is a poor choice
- It relies on asymptotic normality, which is not accurate in finite sample
- A better chioce is "percentile interval"
- First, we stack the sample of bootstrap estimates $\{\hat{\beta}^1, \hat{\beta}^2, ..., \hat{\beta}^R\}$
- We have an empirical distribution of $\hat{\beta}^r$
- The bootstrap $100(1-\alpha)\%$ confidence interval is then: $[q^*_{\alpha/2}, q^*_{1-\alpha/2}]$
- q^* is the quantile of this empirical distribution

Non-parametric Method: Application

- Where to apply non-parametric methods?
- Anything related to estimation of CEF
- Potential outcome framework is non-parametric
- Non-parametric inference in complicated models (Bootstrap)
- If you focus on prediction and fit, but not the structure behind it
  Predict stock price, machine learning, RDD fitting
- We will show these in the following lectures

Final Conclusion

- There are statistical modeling methods other than Linear regression
- Non-parametric methods impose no prior structure, totally data-driven
  - Kernel-based methods: N-W estimator, Local polynomial
  - Series-based methods: Polynomial, Fourier, Spline, Wavelet
- They are very useful in causal inference to directly estimate CEF
- However, they have weaknesses: Not always better to make model more flexible
  - Hard to incorporate restrictions
  - Require large sample size to have accurate estimation
- We will discuss more about it next week
- A semi-parametric model is between non-parametric and parametric

References

Robinson, Peter M. 1988. “Root-N-consistent Semiparametric Regression.” *Econometrica* :931–954.