Quantitative Spatial Economics IV: Dynamic Spatial Model - Part 1[1]

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Overview

Introduction

- We have already discussed the QSEM in the static fashion
- Now let's move forward to a dynamic setting
- Why we need dynamic? What is the value-added here?
- Dynamic nature can amplify or attenuate static effects
- It may give very different policy implications

Introduction

- Forward looking decision is important
- Capital accumulation is important
  - Non-human capital: Machines, Floor space (Housing)
  - Human capital: On-the-job Learning, Children Education
- What is the long-run impact of expressway construction in China?
- What is the long-run impact of enrollment restriction on migrant children?

Introduction

- However, the cost to go to dynamic is very high
- The Dynamic QSEM is difficult to solve
  - In equilibrium, you have $N \times T$ markets and prices to solve
  - You have $N \times T$ or $N^2 \times T$ unobserved fundamentals (productivity, migration cost, trade cost...) to back out
  - Equilibrium equation system becomes much more complicated
  - Dimension of the state space explodes when people move with capital: People's historical movement path becomes critical due to capital flow issue

Introduction

- For investment issue, we have to make assumptions to restrict the state space
- In Caliendo, Dvorkin, and Parro (2019), they drop capital accumulation, only migration is dynamic
- In Kleinman, Liu, and Redding (2023), they separate migration decision from investment decision
  Moving workers vs. Fixing landlords

Introduction

- Three methods to solve the unobserved fundamentals with $N \times T$ markets
  - Dynamic hat-algebra
  - Parameterize fundamentals
  - Invert model to solve fundamentals
- More and more people are preferring the second method after Dingel and Tintelnot (2020), including me
- Overfitting in granular spatial setting $\Rightarrow$ bias-variance tradeoff

Introduction

- We will go through these two papers
- Today, let's start with Caliendo, Dvorkin, and Parro (2019)

Model 1: Caliendo, Dvorkin, and Parro (2019) Dynamic Migration

- In the first model, we consider Caliendo, Dvorkin, and Parro (2019)
- Trade and Labor Market Dynamics: General Equilibrium Analysis of the China Trade Shock
- This will be the most comprehensive spatial model we have learned so far
- It includes various parts we have introduced:
  - Labor migration with friction
  - Goods flow and trade with friction
  - Input-output linkages and sectoral heterogeneity
- Specifically, migration decision is dynamic

Model 1: Caliendo, Dvorkin, and Parro (2019) Dynamic Migration

- Main research question:
  What is the overall effects of China trade shock on U.S. labor markets
- Many studies have investigated this issue like Autor, Dorn, and Hanson (2013)
- But most of them use design-based approach $\Rightarrow$ GE effect?
- Why dynamic?
  - China trade shock is not a one-time shock
  - Migration decision is forward looking
  - People keep moving across time, which changes local labor markets
  - Changes in local labor markets then feed back to change people's moving decision

Model 1: Caliendo, Dvorkin, and Parro (2019) Dynamic Migration

- They find that increased Chinese competition reduces the aggregate manufacturing employment by 0.36 p.p.
- This is about 0.55 million jobs, 16 percent of the observed decline from 2000-2007
- Workers relocate to construction and services sectors
- These two sectors expand thanks to the access to cheaper intermediate from China
- Impact of China shock varies across regions:
  Losses are concentrated, but gains are spreaded
- Overall welfare gain: 0.2%

Model 1 CDP (2019): Settings

CP (2015) firm side + Dynamic migration

- On the firm side (labor demand)
  - Continuum of intermediate goods in different sectors
  - Inputs: Labor, local factors (land), intermediate sectoral goods
  - CRS + perfectly competitive market
- On the household side (labor supply)
  - Forward-looking households with location and labor supply decisions
  - State variables: current location, macro econ state
  - Idiosyncratic shock with T1EV distribution: Dynamic DCM
- Dynamic hat-algebra is used to solve the counterfactual

Model 1 CDP (2019): Settings

- We have N locations and J sectors in the world
- We index i, n as specific locations; j, k as specific sectors
- Time is discrete with $t = 0, 1, 2, ...$

Model 1 CDP (2019): Households

- At t=0, there is a mass L_0^{nj} of households in each location n sector j, either employed or non-employed
- This is the initial labor state of the economy
- For employed households work in sector j region n, they earn wage w_t^{nj} and have a Cobb-Douglas preference:

$$C_t^{nj} = \prod_{k=1}^{J} (c_t^{nj,k})^{\alpha^k}, \quad \sum_{k=1}^{J} \alpha^k = 1 \tag{1}$$

- $c_t^{nj,k}$ is the consumption of sector k goods in market nj at time t

Model 1 CDP (2019): Households

- With this preference, we have a location price index:

$$P_t^n = \prod_{k=1}^{J} (P_t^{nk}/\alpha^k)^{\alpha^k}$$

- P_t^{nk} is the price of sectoral good k in location n
- For non-employed households, they obtain a reserved home production $b^n > 0$
- We denote sector "0" as home production and have $C_t^{n0} = b^n$
- This is an outside option

Model 1 CDP (2019): Households

- Households are forward looking with a discount rate of β
- They make dynamic migration decisions subject to sectoral and spatial mobility costs
- Cost of migrating from nj to ik is: $\tau^{nj,ik} \geq 0$
- It is between region-sector pairs
- Thus, the dimension is $N^2 \times J^2$

Model 1 CDP (2019): Households

- We can express household decision problem formally as:

$$v_t^{nj} = U(C_t^{nj}) + \max_{i,k}\{\beta E(v_{t+1}^{ik}) - \tau^{nj,ik} + \nu \epsilon_t^{ik}\} \qquad (2)$$

- v_t^{nj} is the value flow of being in location n sector j at time t
- This is the "ex post" utility value for each choice nj in the current period t after the preference shock ϵ_t^{ik} is realized
- ϵ_t^{ik} is the idiosyncratic shock on preference for option ik scaled by ν
- Expectation $E(v_{t+1}^{ik})$ is taken w.r.t. future shocks

Model 1 CDP (2019): Households

- We assume that ϵ is i.i.d. and T1EV distributed
- This gives us a closed-form expression of the ex ante utility value:

$$V_t^{nj} = U(C_t^{nj}) + \nu log \left(\sum_i^N \sum_k^J exp(\beta V_{t+1}^{ik} - \tau^{nj,ik})^{1/\nu} \right) \tag{3}$$

- $V_t^{nj} = E[v_t^{nj}]$ is the "ex ante" average value of being in labor market n, j at time t, before the preference shock ϵ_t^{ik} is realized
- It depends on the current-period utility $U(C_t^{nj})$ and the option value of migration in the future
- We can then move it one period forward and replace $E(v_{t+1}^{ik})$ with this expression in ex post utility equation (2)

Model 1 CDP (2019): Households

- Timeline:
  consumption decision at $t = 1$, ϵ_1 realized $\Rightarrow$ v_1 realized, mig decision nj at $t = 1$ $\Rightarrow$ consumption decision at $t = 2$, ϵ_2 realized $\Rightarrow$ v_2 realized, mig decision nj at $t = 2 \Rightarrow$ ...
- Individuals do not know $\epsilon_{t+1}, \epsilon_{t+2}$... when they make decisions at time t
- That is why we have to replace future values by ex ante average values

Model 1 CDP (2019): Households

- We then have a migration share in a Logit style equation:

$$\mu_t^{nj,ik} = \frac{exp(\beta V_{t+1}^{ik} - \tau^{nj,ik})^{1/\nu}}{\sum_m^N \sum_h^J exp(\beta V_{t+1}^{mh} - \tau^{nj,mh})^{1/\nu}} \quad (4)$$

- $\mu_t^{nj,ik}$: Share of households choosing to migrate to ik, among all people originally in nj
- Thus, we have total labor supply in nj at time $t+1$:

$$L_{t+1}^{nj} = \sum_{n'}^{N} \sum_{j'}^{J} \mu_t^{n'j',nj} L_t^{n'j'} \quad (5)$$

- This is the law of motion for macroeconomic states (spatial distribution of labor)

Model 1 CDP (2019): Composite Intermediate

- Production part is static, almost identical to CP (2015)
- The only difference is that we add local factor (land) here
- Intermediate variety goods vs composite intermediate goods
- Composite intermediate: sectoral aggregation of intermediate goods
- Locations trade intermediate goods and then convert them to sectoral composite in local factories
- Sectoral composite then is used for final consumption or production of intermediate varieties
- This is called "Roundabout Production"

Model 1 CDP (2019): Intermediate Production

- First, we have intermediate goods producers in sector j (CD function):

$$q_t^{nj} = z^{nj}(A_t^{nj}(h_t^{nj})^{\xi^n}(l_t^{nj})^{1-\xi^n})^{\gamma^{nj}} \prod_k^J (M_t^{nj,nk})^{\gamma^{nj,nk}}$$

- l_t^{nj} is labor input, h_t^{nj} is local factor input (e.g. land)
- $M_t^{nj,nk}$ is material (composite intermediate) input from sector k to produce j
- z^{nj} and A_t^{nj} are productivities
- CRS: $1 - \gamma^{nj} = \sum_{k=1}^J \gamma^{nj,nk}$

Model 1 CDP (2019): Intermediate Production

- Based on the property of C-D production function, we have the unit price of an input bundle:

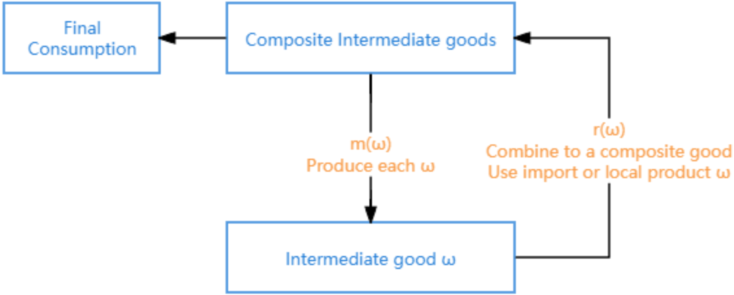
$$x_t^{nj} = B^{nj}((r_t^{nj})^{\xi^n}(w_t^{nj})^{1-\xi^n})^{\gamma^{nj}} \prod_k^J (P_t^{nk})^{\gamma^{nj,nk}} \tag{6}$$

- B^{nj} is a constant
- r_t^{nj} is rental rate for local factor, w_t^{nj} is wage
- P_t^{nk} is price of used materials (composite intermediate)

Model 1 CDP (2019): Composite Intermediate

- We have local sectoral aggregate goods: composite intermediate
- Regions trade intermediate goods (varieties) and then convert them to sectoral composite in local factories
- Sectoral composite then is used for final consumption or production of intermediate varieties

Model 3 CP (2015): Composite Intermediate



Roundabout Production: You trade with intermediates, then combine them together in local factories for consumption and material usage

Model 1 CDP (2019): Composite Intermediate

- We represent iceberg trade costs by $\kappa_t^{nj,ij}$
- Denote $z^j = (z^{1j}, z^{2j}, ..., z^{Nj})$ the vector of productivity by regions
- The price of intermediate variety is:

$$p_t^{nj}(z^j) = \min_i \{ \frac{\kappa_t^{nj,ij} x_t^{ij}}{z^{ij} (A_t^{ij})^{\gamma^{ij}}} \}$$

- We use z^j to index the variety, like ω^j in CP (2015)
- You can think of it as a continuum of goods with a continuum of drawed productivity z
- $p_t^{nj}(z^j)$ is the price of intermediate z^j in sector j used in region j
- Region j shop around all i for each good z^j to find the cheapest one

Model 1 CDP (2019): Composite Intermediate

- Assume we have the following CES production function for composite intermediate:

$$Q_t^{nj} = \left(\int (\tilde{q}_t^{nj}(z^j))^{1-1/\eta^{nj}} d\phi^j(z^j) \right)^{\eta^{nj}/(\eta^{nj}-1)}$$

- Composite/Sectoral intermediate is a CES aggregator of all varieties in this sector
- $\tilde{q}_t^{nj}(z^j)$ is the demand of intermediate variety z^j from the lowest-cost supplier
- $\phi^j(z^j)$ is the cdf for z^j

Model 1 CDP (2019): Composite Intermediate

- Thus, we have a closed-form sectoral goods price:

$$P_t^{nj} = \Gamma^{nj} \left(\sum_i^N (x_t^{ij} \kappa_t^{nj,ij})^{-\theta^j} (A_t^{ij})^{\theta^j \gamma^{ij}} \right)^{-1/\theta^j} \quad (7)$$

- Γ^{nj} is a constant

Model 1 CDP (2019): Composite Intermediate

- Similarly, we have a closed-form expenditure share for composite intermediate firms in market *nj* to spend on intermediate *j* from region *i*

$$\pi^{nj,ij} = \frac{(x_t^{ij}\kappa_t^{nj,ij})^{-\theta^j}(A_t^{ij})^{\theta^j\gamma^{ij}}}{\sum_m^N (x_t^{mj}\kappa_t^{nj,mj})^{-\theta^j}(A_t^{mj})^{\theta^j\gamma^{mj}}} \tag{8}$$

- Trade is positively correlated with exporting regions' productivity
- Negatively correlated with trade cost and production cost

Model 1 CDP (2019): Composite Intermediate

- Comparing trade share and migration share, what do you find?

$$\mu_t^{nj,ik} = \frac{exp(\beta V_{t+1}^{ik} - \tau^{nj,ik})^{1/\nu}}{\sum_m^N \sum_h^J exp(\beta V_{t+1}^{mh} - \tau^{nj,mh})^{1/\nu}}$$

$$\pi^{nj,ij} = \frac{(x_t^{ij} \kappa_t^{nj,ij})^{-\theta^j} (A_t^{ij})^{\theta^j \gamma^{ij}}}{\sum_m^N (x_t^{mj} \kappa_t^{nj,mj})^{-\theta^j} (A_t^{mj})^{\theta^j \gamma^{mj}}}$$

- T2EV (Fréchet) distributed shocks are log-linearized versions of T1EV

Model 1 CDP (2019): Market Clearing and Trade Imbalances

- There are rentiers (landlords) owning local structures, cannot relocate
- They send all rental income to a global portfolio
- Then receive fixed share ι^n from global portfolio with $\sum_n^N \iota^n = 1$ and buy local final goods
- Denote $\chi_t = \sum_i \sum_k r_t^{ik} H^{ik}$ be total revenue in this portfolio
- H^{ik} is the total structure/land supply
- The difference between received income and submitted income generates trade imbalance

Model 1 CDP (2019): Market Clearing and Trade Imbalances

- We do not have investment for landlords, though infrastructure development is crucial
- In Kleinman, Liu, and Redding (2023), we are going to relax this assumption
- Allow the intertemporal dynamic decision for landlords
- This will make the model even more complicated......

Model 1 CDP (2019): Market Clearing and Trade Imbalances

- Let X_t^{nj} be the total expenditure on sector j good in region n
- We have goods market clearing condition as:

$$X_t^{nj} = \underbrace{\sum_k^J \gamma^{nk,nj} \sum_i^N \pi_t^{ik,nk} X_t^{ik}}_{\text{Total production demand}} + \underbrace{\alpha^j \left(\sum_k^J w_t^{nk} L_t^{nk} + \iota^n \chi_t \right)}_{\text{Total consumption demand}} \tag{9}$$

- The first term is the demand from production intermediate usage
- The second term is the demand from final consumption of workers and rentiers
- $\sum_i^N \pi_t^{ik,nk} X_t^{ik}$ is the total expenditure spent on goods from sector k region n

Model 1 CDP (2019): Market Clearing and Trade Imbalances

- Labor market clearing in region n sector j is:

$$L_t^{nj} = \underbrace{\frac{\gamma^{nj}(1-\xi^n)}{w_t^{nj}} \sum_{i}^{N} \pi_t^{ij,nj} X_t^{ij}}_{\text{Labor demand}} \tag{10}$$

- Firms pay a fixed share $\gamma^{nj}(1-\xi^n)$ of revenue on labor usage
- Land market clearing in region n sector j is:

$$H^{nj} = \underbrace{\frac{\gamma^{nj}\xi^n}{r_t^{nj}} \sum_{i}^{N} \pi_t^{ij,nj} X_t^{ij}}_{\text{Structure demand}} \tag{11}$$

- Firms pay a fixed share $\gamma^{nj}\xi^n$ of revenue on land (structure) usage

Model 1 CDP (2019): Equilibrium

- Now we conclude all fundamentals and parameters in this model
- 1. Time-varying fundamentals $\Theta_t \equiv (A_t, \kappa_t)$
  - Productivity: $A_t = \{A_t^{nj}\}_{n=1,j=1}^{N,J}$
  - Trade cost: $\kappa_t = \{\kappa_t^{nj,ij}\}_{n=1,i=1,j=1}^{N,N,J}$
- 2. Constant fundamentals $\bar{\Theta} \equiv (\Upsilon, H, b)$
  - Labor migration cost: $\Upsilon = \{\tau^{nj,ik}\}_{n=1,j=0,i=1,k=0}^{N,J,J,N}$
  - Stock of land/structure: $H = \{H^{nj}\}_{n=1,j=1}^{N,J}$
  - Home production: $b = \{b^n\}_{n=1}^{N}$

Model 1 CDP (2019): Equilibrium

- 3. Parameters (Calibrated)
  - Labor-land composite share in intermediate production: γ^{nj}
  - Labor share in intermediate production: $1 - \xi^{n}$
  - Composite material share in intermediate production: $\gamma^{nk,nj}$
  - Landlord portfolio share: ι^{n}
  - Final consumption share across sectoral goods: α^{j}
  - Discount factor: β
  - Trade and migration elasticity: θ, ν

Model 1 CDP (2019): Equilibrium

- Now we define three layers of the equilibrium in this model
- The first layer is an equilibrium given migration decision in that period

Definition 1 in CDP (2019)

Given $(L_t, \Theta_t, \bar{\Theta})$, a *temporary equilibrium* is a vector of wages $w(L_t, \Theta_t, \bar{\Theta})$ that satisfies the equilibrium conditions of the static subproblem, (6) to (11).

- L_t is the macroeconomic state determined by migration choices
- This is the solution to a static trade model, conditional on migration choices

Model 1 CDP (2019): Equilibrium

- The second layer is a dynamic equilibrium given a path of exogenous fundamentals

Definition 2 in CDP (2019)

Given $(L_0, \{\Theta_t\}_{t=0}^{\infty}, \bar{\Theta})$, a *sequential equilibrium* is a sequence of $\{L_t, \mu_t, V_t, w(L_t, \Theta_t, \bar{\Theta})\}_{t=0}^{\infty}$ that solves the equilibrium conditions (3) to (5) and the temporary equilibrium at each t.

- This is the transition path to the steady state
- Given the *initial* population distribution, and the evolution path of local fundamentals, what will happen in this model
- We replace L_t by L_0 in the information set and endogeneize the dynamic migration decision L_t, μ_t, V_t

Model 1 CDP (2019): Equilibrium

- The third layer is a dynamic equilibrium with constant growth: steady state

Definition 3 in CDP (2019)

A *stationary equilibrium* of the model is a sequential competitive equilibrium such that $\{L_t, \mu_t, V_t, w(L_t, \Theta_t, \bar{\Theta})\}_{t=0}^{\infty}$ are constant for all t.

- Fundamentals are fixed, no aggregate variables change over time

Model 1 CDP (2019): Solving the Model

- Given calibrated parameters and observed data, we have to solve the model
- Compared with the static model, we have more difficulties
- Much more equations in the equilibrium system
- Both data requirement and computation burden are huge
- Can be very hard to invert the model when we have dynamics there
- There are two ways to deal with this issue in CDP (2019)
  - Dynamic Hat Algebra with calibration of shares
  - Parameterize the fundamentals

Model 1 CDP (2019): Hat Algebra

- The first method is "Dynamic Hat Algebra" (DHA) used in CDP (2019)
- This is an extension of the traditional Hat Algebra method in static spatial models
- You can solve the equilibrium responses in changes (hat terms) with changes in fundamentals/economic conditions
- No need to know the levels of the fundamentals

Model 1 CDP (2019): Hat Algebra

- Define $\hat{Y} = Y'/Y$ as the change in variable Y
- We use Hat Algebra to simplify the system by recursively applying the following three rules
- 1. **(Power)** Suppose $Y = X^{\theta}$, then: $\hat{Y} = \hat{X}^{\theta}$
- 2. **(Product)** Suppose $Y = \prod_{i=1}^{N} X_i$, then: $\hat{Y} = \prod_{i=1}^{N} \hat{X}_i$
- 3. **(Sum)** Suppose $Y = \sum_{i=1}^{N} X_i$, then:
  $\hat{Y} = \frac{\sum_i^N X_i'}{\sum_i^N X_i} = \sum_i^N \frac{X_i'}{\sum_{m=1}^N X_m} = \sum_{i=1}^N \frac{X_i}{\sum_{m=1}^N X_m} \frac{X_i'}{X_i} = \sum_{i=1}^N \pi_i \hat{X}_i$ (weighted average in change)

Model 1 CDP (2019): Hat Algebra

- Let's illustrate what is Hat Algebra in a very simple static example
- This example is introduced by Jonathan Dingel in his notes
- Assume we have labor endowment L, productivity shifter χ, trade costs τ, and trade elasticity ϵ
- Endogeneous variables are wage w, income $Y = wL$, and trade flow X_{ij}
- We have the goods market clearing conditions at the equilibrium:

$$w_i L_i = \sum_j^N \lambda_{ij} w_j L_j, \quad \lambda_{ij} = \frac{\chi_i (\tau_{ij} w_i)^{-\epsilon}}{\sum_{l=1}^N \chi_l (\tau_{lj} w_l)^{-\epsilon}}$$

- λ_{ij} is the expenditure share on goods from i

Model 1 CDP (2019): Hat Algebra

- Suppose we have a change of trade cost from τ_{ij} to $\tau_{ij}^{'}$
- We denote the hat term as $\hat{\tau} \equiv \frac{\tau_{ij}^{'}}{\tau_{ij}}$
- Meanwhile, no change of other exogenous fundamentals $\hat{\chi} = \hat{L} = 1$
- Our counterfactual target is: What is the responses of the endogenous variables to this trade cost change?
- Thus, we focus on the "changes" of wage $\hat{w}$, trade share $\hat{\lambda}_{ij}$, and trade flow $\hat{X}_{ij}$
- Levels are not that essential

Model 1 CDP (2019): Hat Algebra

- We write the condition before and after the change:

$$w_i L_i = \sum_j^N \lambda_{ij} w_j L_j, \quad w_i' L_i = \sum_j^N \lambda_{ij}' w_j' L_j = \sum_j^N X_{ij}'$$

- Dividing $w_i' L_i$ by $w_i L_i$ and applying rule 3:

$$\hat{w}_i \hat{L}_i = \sum_j^N \frac{X_{ij}'}{w_i L_i} = \sum_j^N \frac{X_{ij}}{w_i L_i} \hat{X}_{ij} \equiv \sum_j^N \gamma_{ij} \hat{X}_{ij} \tag{12}$$

- $\gamma_{ij} = \frac{X_{ij}}{w_i L_i}$ is the "sales shares" of good from i to j

Model 1 CDP (2019): Hat Algebra

- Similarly, we do the same thing for the gravity equation:

$$\lambda_{ij} = \frac{\chi_i(\tau_{ij} w_i)^{-\epsilon}}{\sum_{l=1}^{N} \chi_l(\tau_{lj} w_l)^{-\epsilon}}, \quad \lambda_{ij}^{'} = \frac{\chi_i^{'}(\tau_{ij}^{'} w_i^{'})^{-\epsilon}}{\sum_{l=1}^{N} \chi_l^{'}(\tau_{lj}^{'} w_l^{'})^{-\epsilon}}$$

- Dividing $\lambda_{ij}^{'}$ by λ_{ij} and applying rules 1, 2 and 3 (for denominator):

$$\hat{\lambda}_{ij} = \frac{\hat{\chi}_i(\hat{\tau}_{ij} \hat{w}_i)^{-\epsilon}}{\sum_{l=1}^{N} \lambda_{lj} \hat{\chi}_l(\hat{\tau}_{lj} \hat{w}_l)^{-\epsilon}} \tag{13}$$

Model 1 CDP (2019): Hat Algebra

- We further assume $\hat{Y} = \hat{X}$ and $\hat{\chi} = \hat{L} = 1$
- Combining equations (12) and (13) together
- We have a system characterizing $\hat{w}_i$ by $\hat{\tau}, \lambda_{ij}, \gamma_{ij}, \epsilon$

$$\hat{w}_i \hat{L}_i = \sum_j^N \gamma_{ij} \hat{X}_{ij} = \sum_j^N \gamma_{ij} \hat{\lambda}_{ij} \hat{w}_j \tag{14}$$

$$\Rightarrow \hat{w}_i = \sum_j^N \frac{\gamma_{ij} \hat{w}_i^{-\epsilon} \hat{\tau}_{ij}^{-\epsilon} \hat{w}_j}{\sum_l^N \lambda_{lj} \hat{w}_l^{-\epsilon} \hat{\tau}_{lj}^{-\epsilon}} \tag{15}$$

- Equation (15) tells us that: We can calculate changes in wages without knowing the levels of fundamentals L and χ
- λ, γ are available in data, ϵ can be estimated $\Rightarrow$ mapping from $\hat{\tau}$ to $\hat{w}$

Model 1 CDP (2019): Dynamic Hat Algebra

- Hat Algebra becomes very useful when you have a large set of unobserved fundamentals, as in the dynamic model case
- Now let's take a look at how does CDP (2019) solve the model by DHA

Model 1 CDP (2019): Dynamic Hat Algebra

- Let's denote $\dot{y}_{t+1} \equiv (y^1_{t+1}/y^1_t, y^2_{t+1}/y^2_t, ...)$, proportional change across periods
- First, we consider solving the original temporary equilibrium at $t+1$ given a change in $\dot{L}_{t+1}, \dot{\Theta}_{t+1}$
- Without needing to know Θ_t and $\bar{\Theta}$

Model 1 CDP (2019): Dynamic Hat Algebra

Proposition 1 in CDP (2019)

Given the allocation of the temporary equilibrium at t, $\{L_t, \pi_t, X_t\}$, the solution to the temporary equilibrium at $t+1$ for a given change in $\dot{L}_{t+1}$ and $\dot{\Theta}_{t+1}$ does not require information on the level of fundamentals at t, Θ_t or $\bar{\Theta}$. In particular, it is obtained as the solution to the following system of nonlinear equations:

$$\dot{x}^{nj}_{t+1} = (\dot{L}^{nj}_{t+1})^{\gamma^{nj}\xi^n} (\dot{w}^{nj}_{t+1})^{\gamma^{nj}} \prod_{k=1}^{J} (\dot{P}^{nk}_{t+1})^{\gamma^{nj,nk}} \tag{16}$$

$$\dot{P}^{nj}_{t+1} = \left(\sum_{i}^{N} \pi^{nj,ij}_t (\dot{x}^{ij}_{t+1} \dot{\kappa}^{nj,ij}_{t+1})^{-\theta^j} (\dot{A}^{ij}_{t+1})^{\theta^j \gamma^{ij}} \right)^{-1/\theta^j} \tag{17}$$

$$\pi^{nj,ij}_{t+1} = \pi^{nj,ij}_t \left(\frac{\dot{x}^{ij}_{t+1} \dot{\kappa}^{nj,ij}_{t+1}}{\dot{P}^{nj}_{t+1}} \right)^{-\theta^j} (\dot{A}^{ij}_{t+1})^{\theta^j \gamma^{ij}} \tag{18}$$

$$X^{nj}_{t+1} = \sum_{k}^{J} \gamma^{nj,nk} \sum_{i}^{N} \pi^{ik,nk}_{t+1} X^{ik}_{t+1} + \alpha^j \left(\sum_{k}^{J} \dot{w}^{nk}_{t+1} \dot{L}^{nk}_{t+1} w^{nk}_t L^{nk}_t + \iota^n \chi_{t+1} \right) \tag{19}$$

$$\dot{w}^{nk}_{t+1} \dot{L}^{nk}_{t+1} w^{nk}_t L^{nk}_t = \gamma^{nj}(1-\xi^n) \sum_{i=1}^{N} \pi^{ij,nj}_{t+1} X^{ij}_{t+1} \tag{20}$$

where $\chi_{t+1} = \sum_{i=1}^{N} \sum_{k=1}^{J} \frac{\xi^i}{1-\xi^i} \dot{w}^{ik}_{t+1} \dot{L}^{ik}_{t+1} w^{ik}_t L^{ik}_t$

Model 1 CDP (2019): Dynamic Hat Algebra

- Bundle cost: (16) comes from (6) and F.O.C.
- Sectoral price: (17) comes from (7)
- Expenditure share: (18) comes from (8)
- Goods market clearing: (19) comes from (9)
- Labor market clearing: (20) comes from (10)

Model 1 CDP (2019): Dynamic Hat Algebra

- Red terms are known values
  - Data moments summarizing all information from unobserved fundamentals' levels
  - Given dynamic aggregate labor movement
- Blue terms are changes in fundamentals, which are determined by your evolution path
- Green terms are "unknowns" needed to be solved

Model 1 CDP (2019): Dynamic Hat Algebra

- Take a look at these five equations
- We can solve $\{\dot{w}^{nj}_{t+1}, \dot{x}^{nj}_{t+1}, \dot{P}^{nj}_{t+1}, \pi^{nj,ij}_{t+1}, X^{nj}_{t+1}\}$ nonlinearly
- Thus, we can solve the changes of endogenous variables without knowing the level of A, κ, τ, H, b

Model 1 CDP (2019): Dynamic Hat Algebra

- Proposition 1 maps changes in fundamentals to changes in endogenous variables in the model, conditional on knowing $\dot{L}$
- That is, we take the dynamic migration decision as given in solving the static equilibrium
- Now we have to add it back and get to solve the dynamic equilibrium

Model 1 CDP (2019): Dynamic Hat Algebra

Definition 5 in CDP (2019)

A converging sequence of changes in fundamentals is such that $\lim_{t \to \infty} \dot{\Theta}_t = 1$

Assumption 3 in CDP (2019)

Agents have logarithmic preferences, $U(C_t^{nj}) \equiv log(C_t^{nj})$.

- We denote $u_t^{nj} \equiv exp(V_t^{nj})$.
- We denote $\dot{\omega}^{nj}(\dot{L}_{t+1}, \dot{\Theta}_{t+1})$ the equilibrium real wages in time differences
- $\dot{\omega}^{nj}(\dot{L}_{t+1}, \dot{\Theta}_{t+1})$ is the solution to Proposition 1.

Model 1 CDP (2019): Dynamic Hat Algebra

Proposition 2 in CDP (2019)

Conditional on an initial allocation of the economy, $(L_0, \pi_0, X_0, \mu_{-1})$, given an anticipated sequence of changes in fundamentals, $\{\dot{\Theta}_t\}_{t=1}^{\infty}$ with $\lim_{t\to\infty} \Theta_t = 1$, the solution to the sequential equilibrium in time differences does not require information on the level of the fundamentals, $\{\Theta_t\}_{t=1}^{\infty}$ or $\bar{\Theta}$, and solves the following system of nonlinear equations:

$$\mu_{t+1}^{nj,ik} = \frac{\mu_t^{nj,ik} (\dot{u}_{t+2}^{ik})^{\beta/\nu}}{\sum_m \sum_h \mu_t^{nj,mh} (\dot{u}_{t+2}^{mh})^{\beta/\nu}} \tag{21}$$

$$\dot{u}_{t+1}^{nj} = \dot{\omega}^{nj}(\dot{L}_{t+1}, \dot{\Theta}_{t+1}) \left(\sum_i \sum_k \mu_t^{nj,ik} (\dot{u}_{t+2}^{ik})^{\beta/\nu} \right)^{\nu} \tag{22}$$

$$L_{t+1}^{nj} = \sum_i \sum_k \mu_t^{ik,nj} L_t^{ik} \tag{23}$$

for all j, n, i, k at each t, where $\{\dot{\omega}^{nj}(\dot{L}_t, \dot{\Theta}_t)\}_{n=1,j=0,t=1}^{N,J,\infty}$ is the solution to the temporary equilibrium given $\{\dot{L}_t, \dot{\Theta}_t\}_{t=1}^{\infty}$

Model 1 CDP (2019): Dynamic Hat Algebra

- Proposition 2 is the key result
- The derivation of these equations are complicated. Refer to Appendix B
- But still, the idea is to using the basic rules of "Hat Algebra"
- The basic idea is still differencing corresponding migration equations

Model 1 CDP (2019): Dynamic Hat Algebra

- We can derive changes in migration choices for a given sequence of fundamental changes with data $(L_0, \pi_0, X_0, \mu_{-1})$ in the initial period
- Solve the model for a given sequence of fundamental changes with data $(L_0, \pi_0, X_0, \mu_{-1})$ in the initial period
- No need to know the fundamentals' levels, since the migration flow contains all information on migration friction levels

Model 1 CDP (2019): Dynamic Hat Algebra

- Algorithm to solve the model with Proposition 1 and 2 is not hard
- It is just a contraction algorithm to solve a fixed point:
  - 1. Guess a path of $\dot{u}$ that ends in $\dot{u} = 1$, the steady state
  - 2. With the entire path of $\dot{u}$ and the initial distributions of μ_{-1}, L_0, we back out the entire path of μ using (21) and L_t using (23) in Proposition 2
  - 3. With the entire path of L_t and the anticipated fundamental changes $\dot{\Theta}$, we back out the path of $\dot{\omega}$ using Proposition 1
  - 4. With the path of $\dot{\omega}, \mu, \dot{u}$, we can update $\dot{u}$ using (22)
  - 5. Repeat step 1-4 until convergence

Model 1 CDP (2019): Dynamic Hat Algebra

- This is it for solving the original equilibrium with anticipated path of fundamentals
- For counterfactuals with unanticipated shocks of fundamentals, it is similar, but even more complicated in notations
- Assume that $y^{'}$ is the value under counterfactual equilibrium
- Define $\dot{y}^{'}_{t+1} \equiv y^{'}_{t+1}/y^{'}_{t}$, the time change under counterfactual equilibrium
- Define $\hat{y}_{t+1} \equiv \dot{y}^{'}_{t+1}/\dot{y}_{t+1}$ the ratio of time change between original and counterfactual equilibria

Model 1 CDP (2019): Dynamic Hat Algebra

Proposition 3 in CDP (2019)

Given a baseline economy, $\{L_t, \mu_{t-1}, \pi_t, X_t\}_{t=0}^{\infty}$, and a counterfactual convergent sequence of changes in fundamentals (relative to the baseline change), $\{\hat{\Theta}_t\}_{t=1}^{\infty}$, solving for the counterfactual sequential equilibrium $\{L_t^{'}, \mu_{t-1}^{'}, \pi_t^{'}, X_t^{'}\}_{t=1}^{\infty}$ does not require information on the baseline fundamentals $(\{\Theta_t\}_{t=0}^{\infty}, \bar{\Theta})$ and solves the following system of nonlinear equations: ......

- Check these equations by yourself in the paper
- Very similar to those in Proposition 1 and 2

Model 1 CDP (2019): Dynamic Hat Algebra

- Dynamic hat algebra is the dynamic version of hat algebra
- Thus, it has the same issue of hat algebra: overfitting
- Dingel and Tintelnot (2020) discuss this issue in details
- The idea of dynamic hat algebra is to match theoretical shares of migration and trade to the empirical counterparts
- When using Propositions 1, 2, and 3, you have to derive π, X and μ from data
- But sampling data can always be noisy
- Exact match means matching not only signal, but also noise

Model 1 CDP (2019): Parameterize Fundamentals

- Thus, we introduce the second method to deal with the huge set of unobserved fundamentals
- We cannot observe them, but we can assume they are functions of observed variables
- In traditional gravity equation, we assume trade cost to be a function of distance, tariff, and other policy trade barriers
- In traditional urban model, we assume commuting cost to be a function of distance, transportation infrastructure, and Hukou policy
- We do not need to capture the whole data pattern
- We regularize the data by giving it a "structure", and extract the signal

Model 1 CDP (2019): Parameterize Fundamentals

- Here is a possible way to parameterize this model
- Let's run a regression for empirical trade share $\tilde{\pi}^{nj,ij}$ from data:

$$\tilde{\pi}^{nj,ij} = \underbrace{\beta_0 + \beta_1 distance_{ni} + \beta_2 Tariff_{nj,ij}}_{\text{Signal}} + \underbrace{\epsilon_{nj,ij}}_{\text{Noise}}$$

- We take the signal part to be our trade cost τ: $\tau = \hat{\beta}_0 + \hat{\beta}_1 distance_{ni} + \hat{\beta}_2 Tariff_{nj,ij}$
- We admit the uncertainty/noise and accept it
- Bias-variance tradeoff!

Model 1 CDP (2019): Parameterize Fundamentals

- With this parameterization of fundamentals, we back out their levels
- We can easily plug them back and solve the model with a contraction algorithm
- I will not go through the details of the algorithm
- Please refer to Professor Ma Lin's notes

Model 1 CDP (2019): Examples of Counterfactual Questions

Counterfactual 1: Dynamics with constant fundamentals

- Given initial allocation, how would the economy evolve over time?
- We can answer this question using Proposition 2, where fundamentals do not change over time
- This is solving a transition path: not that much a counterfactual
- Data requirement: initial allocation $(L_0, \pi_0, X_0, \mu_{-1})$

Model 1 CDP (2019): Examples of Counterfactual Questions

Counterfactual 2: Unexpected hypothetical changes in fundamentals

- A subset of fundamentals change unexpectedly by agents
- First, we solve the evolution of the economy without the unexpected change using Proposition 2
- Here we use constant/true fundamental changes
- Then, we use the results of this baseline economy to calculate the economy with unexpected change using Proposition 3
- Data requirement: initial allocation $(L_0, \pi_0, X_0, \mu_{-1})$, fundamental changes $\dot{\Theta}$

Model 1 CDP (2019): Examples of Counterfactual Questions

Counterfactual 3: Unexpected actual changes in fundamentals

- This is the question they answer in the main paper: what was the effect of the actual China shock?
- The process is similar to the last case
- But here we have an actual change rather than hypothetical change
- Thus, we must measure the China shock in the real world

Model 1 CDP (2019): Economy without China Shock

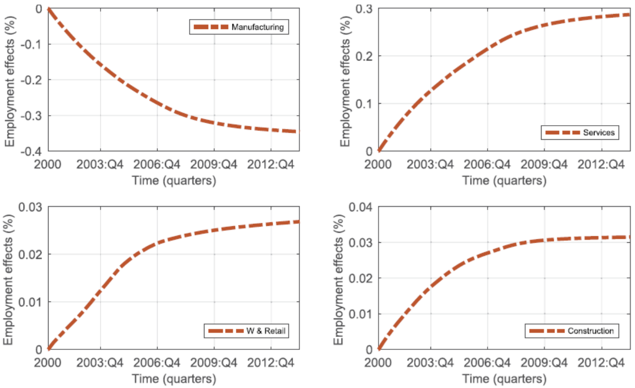


FIGURE 1.—The effect of the China shock on employment shares. Note: The figure presents the effects of the China shock measured as the change in employment shares by sector (manufacturing, services, wholesale and retail, and construction) over total employment between the economy with all fundamentals changing as in the data and the economy with all fundamentals changing except for the estimated sectoral changes in productivities in China (the economy without the China shock).

Model 1 CDP (2019): Economy without China Shock

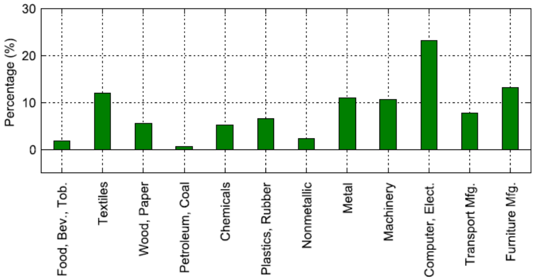


FIGURE 2.—Manufacturing employment declines due to the China trade shock (percent of total). Note: The figure presents the contribution of each manufacturing industry to the total reduction in the manufacturing employment due to the China shock.

Model 1 CDP (2019): Economy without China Shock

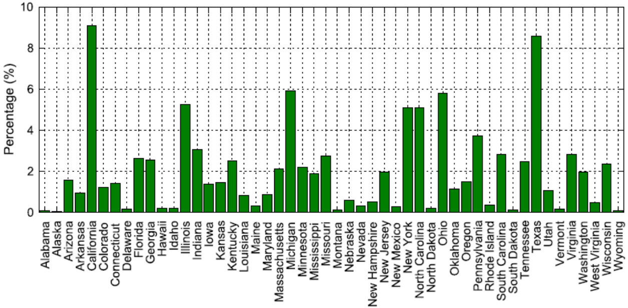


FIGURE 3.—Regional contribution to U.S. aggregate manufacturing employment decline (percent). Note: The figure presents the contribution of each state to the total reduction in manufacturing sector employment due to the China shock.

Model 1 CDP (2019): Economy without China Shock

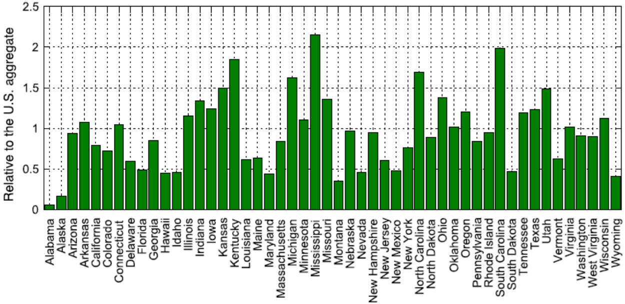


FIGURE 4.—Regional contribution to U.S. aggregate manufacturing employment decline, normalized by regional employment share. Note: The figure presents the contribution of each state to the U.S. aggregate reduction in manufacturing sector employment due to the China shock, normalized by the employment of each state relative to the U.S. aggregate employment.

Model 1 CDP (2019): Economy without China Shock

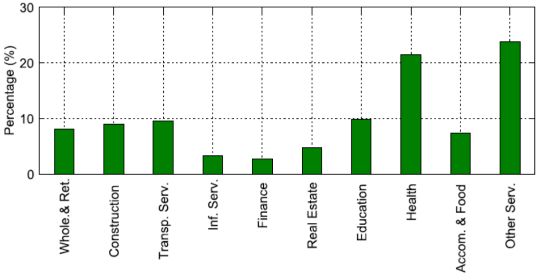


FIGURE 5.—Non-manufacturing employment increases due to the China trade shock (percent of total). Note: The figure presents the contribution of each non-manufacturing sector to the total increase in non-manufacturing employment due to the China shock.

Model 1 CDP (2019): Economy without China Shock

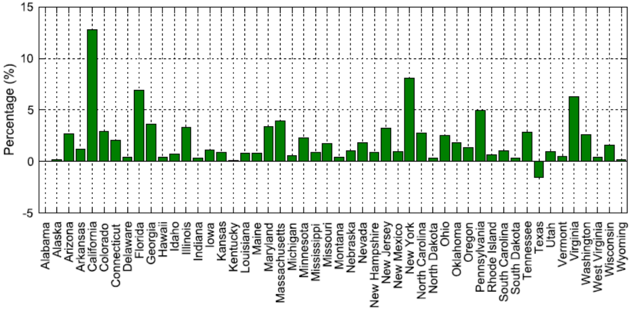


FIGURE 6.—Regional contribution to U.S. aggregate non-manufacturing employment increase (percent). Note: The figure presents the contribution of each state to the total rise in non-manufacturing employment due to the China shock.

Model 1 CDP (2019): Economy without China Shock

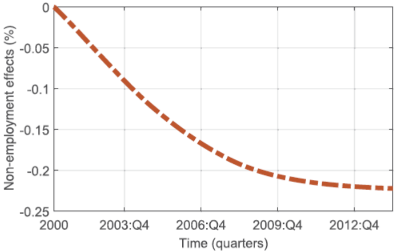


FIGURE 9.—The effect of the China shock on non-employment shares. Note: The figure presents the effects of the China shock, measured as the difference in the non-employment shares between the economy with all fundamentals changing as in the data and the economy with all fundamentals changing except for the estimated sectoral changes in productivities in China (the economy without the China shock).

Model 1 CDP (2019): Economy without China Shock

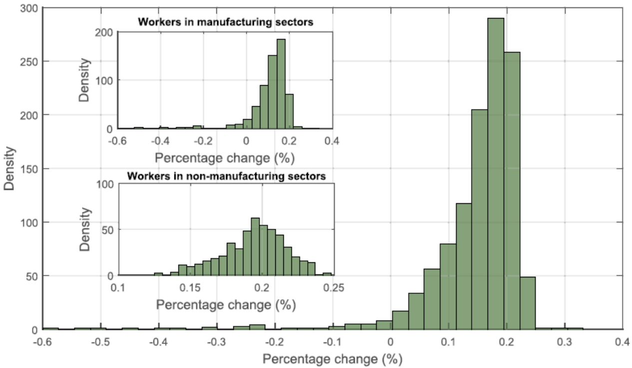


FIGURE 10.—Welfare effects of the China shock across U.S. labor markets. Note: The figure presents the change in welfare across all labor markets (central figure), for workers in manufacturing sectors (top-left panel), and for workers in non-manufacturing sectors (bottom-left panel) as a consequence of the China shock. The largest and smallest 1 percentile are excluded in each figure. The percentage change in welfare is measured in terms of consumption equivalent variation.

Model 1 CDP (2019): Economy without China Shock

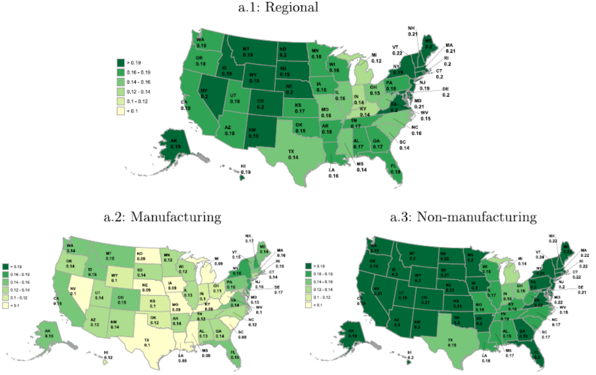


Figure 11: Regional welfare effects (percent)

FIGURE 11.—Regional welfare effects (percent). Note: The figure presents the welfare effects across states in the United States. Panel a.1 shows the regional effects in each state, panel a.2 presents the manufacturing welfare effects in each state, and panel a.3 presents the welfare effects in the non-manufacturing sectors in each state. We aggregate welfare across labor markets within a state using employment shares for the initial year.

Model 1 CDP (2019): Economy without China Shock

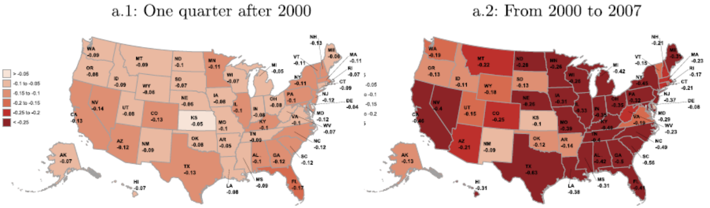


FIGURE 12.—Regional real wage changes in the manufacturing sector (percent). Note: The figure presents the change in real wages in the manufacturing sector across U.S. states. Panel a.1 presents the change in real wages at impact, one quarter after the China shock started. Panel a.2 presents the change in real wages from 2000 to 2007, during the entire period of the China shock. We aggregate the changes in real wages across labor markets within a state using employment shares for the initial year.

Conclusion

- In the next lecture, we will introduce Kleinman, Liu, and Redding (2023)
- Dynamic Spatial Equilibrium Model is much much harder to solve compared with the static one
- But as you can see from the lectures, they share similar modeling patterns and solving techniques
- I really hope you guys to do some work on this
- Of course, start from replicating or mimicing the model in a China topic

Conclusion

- It is a truly developing area with many questions unsettled:
  - Agents can make both migration and saving decisions
  - Saving and financial flows across regions
  - Knowledge spillover and idea flows
  - Government bond and debt, strategic interaction
  - Aggregate uncertainty

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