Quantitative Spatial Economics II: Spatial Model with Trade

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Overview

Introduction

- We have already discussed the QSEM with migration
- This is the flow of people
- Now let's consider the flow of goods: trade
- They are very similar in model settings
  - Fréchet distribution: Productivity for trade vs. Preference for migration
  - Finding goods with lowest price vs. Finding places with highest utility
  - Probabilistic trade vs Probabilistic migration

Introduction

- I will only go through very basic ideas in one week
- It is extremely helpful to take Prof.Deng's course
- You may also read Prof.Wang's lecture notes
- They will introduce trade model very carefully

Introduction

- The important question in trade is:
  - Why do people consume goods from different countries?
  - Why countries import/export different goods from/to other countries?
- This is the mirror question as in migration case (why do people migrate?)
- Today we will start from the simplest Armington model, to EK (2002) and CP (2015)

Model 1: Armington Model

- Let's start from Armington model (Armington, 1969)
- In this model, the reason why there is trade is simple and straightforward
- Because consumers need them due to the form of the utility function
- This is more like a mechanical explanation
- We can learn how does CES demand system work, which will be employed repeatedly in QSEM

Model 1 Armington: Consumer

- Consider there are N countries in the world, each denoted by i, n, k
- For consumers in country n, we assume that they have the following CES utility function:

$$U_n = \left[\sum_{i=1}^{N} C_{in}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 1 \tag{1}$$

- C_{in} is the consumption of consumers in country n on goods from country i
- σ is the elasticity of substitution among goods from different countries

Model 1 Armington: Consumer

- Denote that we have national total income Y_n, national total expenditure X_n
- We can then set up the Lagrangian function to solve this optimization problem:

$$\max_{C_{in}} U_n = \left[\sum_{i=1}^{N} C_{in}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

$$\text{s.t.} \sum_i P_{in} C_{in} = Y_n$$

- This is a typical CES demand system

Model 1 Armington: Consumer

- It turns out that we have the following conclusions
- First, the expenditure on goods from country i can be expressed as:

$$X_{in} = P_{in}^{1-\sigma} P_n^{\sigma-1} X_n \tag{2}$$

$$P_n = \left[\sum_{i=1}^{N} P_{in}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \tag{3}$$

- P_n is called Dixit-Stiglitz Price Index
- It illustrates the overall price level of the home country n
- This expression shows that people spend a share of their total income to goods from each country
- Cheaper goods get larger market share

Model 1 Armington: Consumer

- Second, we can express the indirect utility as:

$$U_n = \frac{Y_n}{P_n} \tag{4}$$

- The welfare of consumers from country n is equal to the real income
- That is, it depends on the income and the overall price index

Model 1 Armington: Producer

- Now let's consider the producer
- Denote w_i as wage, $T_i^{\frac{\sigma-1}{1}}$ as productivity in country i
- The unit cost of goods from i can be expressed as:

$$c_i = \frac{w_i}{T_i^{\frac{\sigma-1}{1}}} \tag{5}$$

- Additionally, we consider trade cost τ_{in}
- In a competitive market, we have price equals cost, which gives us:

$$P_{in} = \frac{w_i \tau_{in}}{T_i^{\frac{\sigma-1}{1}}} \tag{6}$$

Model 1 Armington: Producer

- We plug (6) to (2) and have the share of consumption on goods from i:

$$\frac{X_{in}}{X_n} = \frac{T_i w_i^{-\theta} \tau_{in}^{-\theta}}{\sum_k^N T_k w_k^{-\theta} \tau_{kn}^{-\theta}} \tag{7}$$

- This is the gravity equation for goods flow
- $\theta = \sigma - 1 > 0$ is the trade elasticity
- It shows that trade flow is negatively related to trade cost and production cost, but positively related to productivity

Model 1 Armington: Pros and Cons

- Armington model is very simple and straightforward
- We can directly derive the gravity equation for trade flow
- It is also easy to incorporate data to this model and do counterfactual works
- However, it lacks a micro foundation
- Why do people consume goods from different countries?
- Because they have them in the utility function
- This is more like no explanation at all, a way to bypass the discussion

Model 2: Eaton and Kortum (2002) Probabilistic Trade

- We then go to the original probabilistic trade model in Eaton and Kortum (2002)
- Different from Armington model, it is built on a micro foundation
- Consumers choose goods from different countries because different countries have price advantages in different goods
- Productivity distribution $\Rightarrow$ Price distribution $\Rightarrow$ Probabilistic trade/expenditure

Model 2 EK (2002): Settings

- EK (2002) is the first model to introduce probabilistic trade
- Based on perfect competition and all other main assumptions in neo-classical trade model
- Trade flows across multi-country, Ricardian style
- Derive gravity equation with micro foundation
- Easy to incorporate data

Model 2 EK (2002): Settings

- Assume that we have N countries, a continuum of good $j \in [0, 1]$
- For country i, its efficiency in producing j is $z_i(j)$
- There is an iceberg trade d_{ni} for moving goods from i to n
- Input cost in i is c_i; Constant returns to scale
- Thus, we have the cost of producing a unit of good j in country i to be $c_i/z_i(j)$
- Delivering it from i to n costs:

$$p_{ni}(j) = \frac{c_i}{z_i(j)} d_{ni}$$

- Perfect competitive market $\Rightarrow$ price equals cost

Model 2 EK (2002): Settings

- One important difference compared with Armington model is that we introduce a continuum of good
- You will see very soon that this is essential for probabilistic trade
- By this way, we can assume a continuous distribution of the productivity/prices

Model 2 EK (2002): Consumer

- Consumers in country n maximize a CES utility:

$$U = \left[\int_0^1 Q(j)^{(\sigma-1)/\sigma} dj \right]^{\sigma/(\sigma-1)}$$

- They shop around the world across i for the best deal:

$$p_n(j) = \min\{p_{ni}(j); i = 1, 2, ..., N\}$$

- $p_n(j)$ is the actual price paid by the consumer for variety j
- We no longer assume that people need goods from different countries because utility function tells them to do that
- But now we give it a micro foundation: price minimization

Model 2 EK (2002): Technology

- We assume that country i has production efficiency $z_i(j)$ on good j
- It is a random variable: this is the key to derive the probabilistic trade

$$F_i(z) = e^{-T_i z^{-\theta}}$$

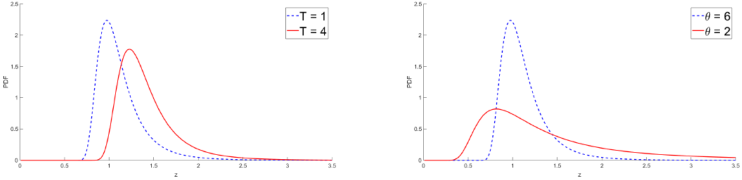
- $F_i(z)$ is the probability for good j in country i to have an efficiency lower than z
- It is Fréchet distributed

Model 2 EK (2002): Technology

- Just assume that you have many types of goods
- You assign an efficiency of production to each of them from this distribution
- Thus, you are good at producing some of them, but bad at others
- Therefore, consumers in country n have a distribution of prices for product j from different countries i: $p_{ni}(j)$
- Although you have a deterministic choice for each j
  I choose Coke from U.S., Benz from Germany
- When aggregating over the continuum, you have a "proportion" of goods coming from a specific country

Model 2 EK (2002): Technology

- T_i: Mean, absolute advantage
- θ: Dispersion, comparative advantage



(a) $\theta = 6$ (b) $T = 1$

Model 2 EK (2002): Technology

- The derivation of the goods flow equation is similar to what we have done in the last lecture
- The distribution of price for goods from i sold in country n is:

$$G_{in}(p) = Pr(p_{in}(j) \leq p) = F\left(z_i(j) \geq \frac{c_i d_{ni}}{p}\right)$$
$$= 1 - exp\{-T_i(c_i d_{ni})^{-\theta} p^{\theta}\} = 1 - e^{-\Phi_{ni} p^{\theta}}$$

- We can then calculate the pdf as the derivative of this:

$$g_{in}(p) = \frac{\partial G_{in}(p)}{\partial p} = \Phi_{ni} \theta p^{\theta - 1} e^{-\Phi_{ni} p^{\theta}}$$

Model 2 EK (2002): Technology

- Therefore, the distribution of the minimum price within the goods choice set for consumer in country n can be expressed as:

$$
\begin{aligned}
Pr(p_n(j)) &= Pr(\min_i p_{in}(j) \le p) = 1 - Pr(\min_i p_{in}(j) > p) \\
&= 1 - \prod_i^N [1 - Pr(p_{in}(j) \le p)] \\
&= 1 - exp(\sum_i^N -T_i(c_i d_{ni})^{-\theta} p^{\theta}) = 1 - e^{-\Phi_n p^{\theta}}
\end{aligned}
$$

Model 2 EK (2002): Technology

- Now we calculate the probability for producing country i to offer the lowest price of good j in market country n

$$\pi_{ni} = Pr(p_{in}(j) \leq \min\{p_{sn}(j); s \neq i\}) = \int_0^\infty \prod_{s \neq i} [1 - G_{sn}(p)] g_{in}(p) dp$$

$$= \int_0^\infty \prod_{s \neq i} (e^{-\Phi_{ni} p^\theta})(\Phi_{ni} \theta p^{\theta - 1} e^{-\Phi_{ni} p^\theta}) dp = \frac{\Phi_{in}}{\Phi_n}$$

Model 2 EK (2002): Technology

- Therefore, based on the property of the Fréchet distribution, we have:

$$\pi_{ni} = \frac{T_i(c_i d_{ni})^{-\theta}}{\sum_{s=1}^{N} T_s(c_s d_{ns})^{-\theta}} = \frac{X_{ni}}{X_n}$$

- π_{ni} is the probability for producing country i to offer the lowest price of good j in market country n
- It is also the expenditure fraction for country n's consumer to spend on country i's good
- Positively related to overall productivity in i
- Negatively related to production bundle cost and trade cost

Model 2 EK (2002): Technology

- The price index can also be derived as:

$$p_n = \gamma \Phi_n^{-1/\theta}$$

- where $\Phi_n = \sum_{r=1}^{N} T_r (c_r d_{nr})^{-\theta}$
- γ is a constant
- This is the overall price level in country n
- Similar to the overall utility level in the migration case
- Price index is in fact closely related to the welfare $\Rightarrow$ Real wage

Model 2 EK (2002): Technology

- This is so called "probabilistic trade"
- Just like probabilistic migration
- In migration case, you choose your living place
  - In a set of locations
  - With an idiosyncratic shock in your preference
  - To find the location with the highest utility
- In trade case, you choose a good
  - In a set of goods from different source countries
  - With an idiosyncratic shock in source country productivity
  - To find the good with the lowest cost
- Individual choice is deterministic ex post
- But aggregated choice is probabilistic/proportional

Model 3: Caliendo and Parro (2015) Trade with Sectoral Linkage

- In the third model, we consider Caliendo and Parro (2015)
- This is still a static spatial model with trade
- Building on EK (2002), we add sectoral heterogeneity and input-output network in this setting
- There are different sectors producing intermediate goods
- The production process takes labor and intermediate goods as inputs

Model 3: Caliendo and Parro (2015) Trade with Sectoral Linkage

- We have a production network with sectoral linkages
- This means that policies directly affecting one sector will spillover to other related sectors
- For example, U.S. imposes a tariff on China's battery
- Then, it will affect Tesla who uses battery to build their EVs
- This is important, and hard to be investigated in pure design-based approach
- Because in settings with spillovers, SUTVA is violated

Model 3 CP (2015): Settings

- The basic settings of CP (2015) are very similar to EK (2002)
- We have N countries, J sectors
- i, n denote countries, j, k denote sectors
- Production uses labor and intermediate goods as inputs
- All markets are perfectly competitive, labor is mobile across sectors

Model 3 CP (2015): Households

- There are L_n households in each country, maximizing utility by consuming final goods:

$$u(C_n) = \prod_{j=1}^{J} (C_n^j)^{\alpha_n^j}, \quad \sum_{j=1}^{J} \alpha_n^j = 1$$

- C-D utility with final goods in each sector j: new dimension
- C-D utility is a special case for CES utility when the elasticity of substitution is fixed at 1

Model 3 CP (2015): Intermediate

- A continuum of intermediate goods $\omega^j \in [0,1]$ is produced in each sector j
- Two types of inputs: labor l_n^j, composite intermediate goods $m_n^{k,j}$ from all sectors k:

$$q_n^j(\omega^j) = z_n^j(\omega^j)[l_n^j(\omega^j)]^{\gamma_n^j} \prod_{k=1}^{J} [m_n^{k,j}(\omega^j)]^{\gamma_n^{k,j}}$$

- $q_n^j(\omega^j)$ is the production quantity of good ω^j in sector j in country n
- $z_n^j(\omega^j)$ is the production efficiency, Fréchet distributed
- $m_n^{k,j}(\omega^j)$ is the composite intermediate good of sector k used in producing good ω^j
- Materials used in production is not "variety" ω^k, but its sectoral aggregator

Model 3 CP (2015): Intermediate

- As before, CRS and perfect competition give us firm price at unit cost: $c_n^j / z_n^j(\omega^j)$
- With the property of C-D production function, we have:

$$c_n^j = \Upsilon_n^j w_n^{\gamma_n^j} \prod_{k=1}^{J} (P_n^k)^{\gamma_n^{k,j}}$$

- w_n: wage rate in country n
- P_n^k: price of a composite intermediate good in sector k
- Υ_n^j: constant
- Basically, you just replace factor quantities by factor prices in the production function

Model 3 CP (2015): Composite Intermediate

- Composite intermediate good in sector j is a combination of all intermediates in this sector
- Producers of composite intermediate supply Q_n^j
- At minimum cost by purchasing intermediate goods ω^j from the lowest cost suppliers across countries

$$Q_n^j = [\int r_n^j(\omega^j)^{1-1/\sigma^j} d\omega^j]^{\sigma^j/(\sigma^j-1)}$$

- $r_n^j(\omega^j)$ is the input demand of intermediate ω^j from lowest cost supplier
- Countries import/export intermediate, but not composite intermediate!

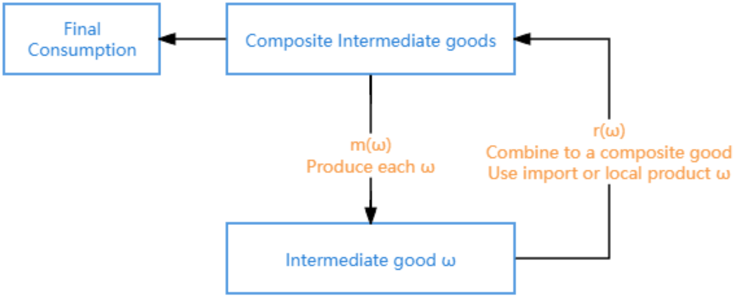
Model 3 CP (2015): Composite Intermediate

- By solving the F.O.C., we have the following demand function for intermediate ω^j:

$$r_n^j(\omega^j) = (\frac{p_n^j(\omega^j)}{P_n^j})^{-\sigma^j} Q_n^j$$

- $P_n^j = [\int p_n^j(\omega^j)^{1-\sigma^j} d\omega^j]^{1/(1-\sigma^j)}$, overall input price index in sector j
- Demand of one single good is positively related to the production quantity Q_n^j and the overall input price P_n^j
- And negatively related to its own price $p_n^j(\omega^j)$
- Composite intermediate is used as
  - Materials for production of intermediate ω as $m_n^{k,j}(\omega^j)$
  - Final goods consumption

Model 3 CP (2015): Composite Intermediate



Roundabout Production: You trade with intermediates, then combine them together in local factories for consumption and material usage

Model 3 CP (2015): Intermediate Trade Costs and Prices

- We assume an iceberg trade cost κ^j_{ni} for each pair of countries and each sector
- Therefore, the price of intermediate good ω^j in country n is:

$$p^j_n(\omega^j) = \min_i \{ \frac{c^j_i \kappa^j_{ni}}{z^j_i(\omega^j)} \}$$

- We assume that $z^j_i(\omega^j)$ follows a Fréchet distribution with location parameter λ^j_n and shape parameter θ^j
- Productivity varies across both countries and sectors
- Why do we need this heterogeneity across sectors?
- Because U.S. is good at producing aircraft, but not T-shirt

Model 3 CP (2015): Intermediate Trade Costs and Prices

- By the property of the Fréchet distribution, we have the price of the composite intermediate good:

$$P_n^j = A^j [\sum_{i=1}^{N} \lambda_i^j (c_i^j \kappa_{ni}^j)^{-\theta^j}]^{-1/\theta^j}$$

- A^j is a constant
- Thus, consumer price index in country n is:

$$P_n = \prod_{j=1}^{J} (P_n^j / \alpha_n^j)^{\alpha_n^j}$$

Model 3 CP (2015): Expenditure Shares

- Finally, we can calculate the expenditure shares of country n on sectoral goods j from country i:

$$\pi_{ni}^j = \frac{\lambda_i^j [c_i^j \kappa_{ni}^j]^{-\theta^j}}{\sum_{h=1}^N \lambda_h^j [c_h^j \kappa_{nh}^j]^{-\theta^j}}$$

- How can tariff in one sector affect trade in another sector?
- Because $c_n^j = \Upsilon_n^j w_n^{\gamma_n^j} \prod_{k=1}^J (P_n^k)^{\gamma_n^{k,j}}$, cost of producing j depends on prices of composite intermediate in all sector k
- Tariff in sector k affects price in this sector P_n^k, and therefore π_{ni}^j

Conclusion

- Today we learn how to model goods trade in spatial models
- The old-fashion Armington model is simple but with no micro foundation
- It assumes that the utility has to combine goods from different countries
- Consumers just need them because the utility told them so
- It gives us a clear gravity equation of trade

Conclusion

- EK model introduces micro foundation for trade in quantitative model
- It is very similar to the migration case
- It assume that consumers shop around goods globally by choosing the cheapest one
- Distribution in productivity $\Rightarrow$ Distribution in price $\Rightarrow$ Probabilistic trade

Conclusion

- CP model further extends EK model by introducing sectoral heterogeneity and production network
- It constructs a roundabout production system with intermediate and composite intermediate goods
- But the basic idea is still the same:
  Distribution in productivity $\Rightarrow$ Distribution in price $\Rightarrow$ Probabilistic trade

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