Quantitative Spatial Economics IV: Dynamic Spatial Model - Part 2[1]

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Overview

Model 2: Kleinman, Liu, and Redding (2023) Dynamic Migration + Investment

- In the second model, we consider Kleinman, Liu, and Redding (2023)
- Compared with CDP (2019), KLR (2023) takes one step further
- In CDP (2019), landlords are passive, no investment on local structure
- In KLR (2023) landlords become forward-looking agents
- They make intertemporal investment decision to improve capital used in production

Model 2: Kleinman, Liu, and Redding (2023) Dynamic Migration + Investment

- In this paper, they have:
  - Forward-looking households making consumption and migration decisions
  - Households are hand-to-mouth, no saving for them
  - Armington style trade model (isomorphic to EK style in this setting, See Supplement S.3.1)
  - Frictional migration and frictional trade
  - Production determined by labor, capital, and technology
  - Forward-looking landlords making capital investment decision
- Life becomes harder and harder......
- But this is academia. Keep yourself at the frontier!

Model 2: Kleinman, Liu, and Redding (2023) Dynamic Migration + Investment

- Using this dynamic spatial GE model with investment
- They investigate the determinants of income convergence across U.S. states from 1965-2015
- And the persistent and heterogeneous impact of local productivity and amenity shocks
- They show that the decline in the rate of income convergence is driven by initial conditions, but not changes of shocks to fundamentals
- Both capital and labor dynamics are important for capturing this decline
- They find very slow convergence to the steady state

Model 2: Kleinman, Liu, and Redding (2023) Dynamic Migration + Investment

- This model can be generalized to the following components
  - Shocks to trade and migration costs
  - Agglomeration forces
  - Multiple sectors
  - Input-output linkages
  - Residential capital (Housing market)
  - Non-employment

Model 2: Kleinman, Liu, and Redding (2023) Dynamic Migration + Investment

- They have a very rich document for this paper
- Main paper (40 pages) + Appendix (24 pages) + Supplement (146 pages)
- They show how to solve the model in three different method
  - Extended dynamic hat-algebra from CDP (2019)
  - Solve unobserved fundamentals in level
  - Linearize the system and use spectral analysis
- We will focus on the first two methods
- The third one is important for analyzing convergence rate

Model 2 KLR (2023): Settings

- Now we go to the settings of this model
- Consider we have many locations indexed by $i \in \{1, ..., N\}$
- We also have many periods indexed by t
- Endogenous variables in each location: population l_{it} and capital stock k_{it}
- Location fundamentals: productivity z_{it}, amenities b_{it}, trade costs τ_{nit}, migration costs κ_{nit}

Model 2 KLR (2023): Production

- At the beginning of each period t, we have a mass of workers l_{it} and a capital stock k_{it} in each location
- We have a CRS Cobb-Douglas production function:

$$y_{it} = z_{it} \left(\frac{l_{it}}{\mu} \right)^{\mu} \left(\frac{k_{it}}{1-\mu} \right)^{1-\mu} \tag{1}$$

- z_{it} is productivity
- Input factors: labor l_{it}, capital k_{it}
- Perfect competitive market
- No input-output linkage or production network in this model

Model 2 KLR (2023): Production

- Assume an iceberg trade cost τ_{nit} by shipping goods from i to n
- Unit cost function can be easily derived using FOCs as:

$$p_{nit} = \frac{\tau_{nit} w_{it}^{\mu} r_{it}^{1-\mu}}{z_{it}} \tag{2}$$

- p_{nit} is the price for consumers in n to pay for goods from i
- w_{it} and r_{it} are wages and rental rates in i

Model 2 KLR (2023): Workers

- We have a CES preference as in the Armington model
- Workers in n consume a variety of goods from different regions $i = 1, ..., N$:

$$u_{nt}^{w} = b_{nt} c_{nt}, \quad c_{nt} = \left[\sum_{i=1}^{N} (c_{nit}^{w})^{\frac{\theta}{\theta+1}} \right]^{\frac{\theta+1}{\theta}} \theta = \sigma - 1 \tag{3}$$

- Superscript w denotes worker (differ from landlord)
- c_{nt} is local consumption index
- b_{nt} is local amenity
- This is a simple way to rationalize trade flows without micro foundation
- You need goods from all countries because the preference told you so

Model 2 KLR (2023): Workers

- Thus, we have the following indirect utility function using FOCs:

$$u_{nt}^{w} = \frac{b_{nt} w_{nt}}{p_{nt}}, \quad p_{nt} = \left[\sum_{i=1}^{N} p_{nit}^{-\theta} \right] \tag{4}$$

- p_{nt} is the price index in location n
- p_{nt} is also a measure of the welfare in location n

Model 2 KLR (2023): Landlords

- Landlords choose their consumption and investment to maximize utility

$$v_{it}^{k} = E_t \sum_{s=0}^{\infty} \beta^{t+s} \frac{(c_{it+s}^{k})^{1-1/\psi}}{1-1/\psi} \tag{5}$$

$$r_{it} k_{it} = p_{it}(c_{it}^{k} + k_{it+1} - (1-\delta) k_{it}) \tag{6}$$

- Landlords have a utility function related to the consumption index c_{it+s}^{k}, which has the same Armington structure as for workers
- They have a forward-looking consumption-investment tradeoff
- (6) is the budget constraint for the landlords
- δ is the capital depreciation rate

Model 2 KLR (2023): Landlords

- We denote $R_{it} \equiv 1 - \delta + r_{it}/p_{it}$ as the gross return on capital
- By guess and verify, we have landlords' optimal saving decision

Lemma 1 in KLR (2023)

The optimal consumption of location i's landlords satisfies $c_{it} = \varsigma_{it} R_{it} k_{it}$, where ς_{it} is defined recursively as:

$$\varsigma_{it}^{-1} = 1 + \beta^{\psi} (E_t[R_{it+1}^{\frac{\psi-1}{\psi}} \varsigma_{it+1}^{-\frac{1}{\psi}}])^{\psi}$$

Landlords' optimal saving and investment decisions satisfy $k_{it+1} = (1 - \varsigma_{it}) R_{it} k_{it}$

Model 2 KLR (2023): Landlords

- Landlords have a linear saving rate, depending on:
  - Future returns $R_{it+1}, R_{it+2}\ldots$
  - Discount rate β
  - Intertemporal elasticity of substitution ψ
- When we have a log-utility, landlords have a constant saving rate β $k_{it+1} = \beta R_{it} k_{it}$
- Landlords and capital are geographically immobile

Model 2 KLR (2023): Worker Migration

- Workers in i make their migration decisions to maximize their flow of utility

$$V_{it}^{w} = lnu_{it}^{w} + \max_{\{g\}_1^N}\{\beta E_t[V_{gt+1}^{w}] - \kappa_{git} + \rho\epsilon_{gt}\} \tag{7}$$

- $\{g\}_1^N$ is the set of potential destinations
- κ is the migration cost from i to g
- ϵ_{gt} are preference shocks drawn from a T1EV distribution

Model 2 KLR (2023): Market Clearing

- Goods market clearing means income equals expenditure in each location:

$$w_{it} l_{it} + r_{it} k_{it} = \sum_{n=1}^{N} S_{nit} (w_{nt} l_{nt} + r_{nt} k_{nt}) \tag{8}$$

- $w_{it} l_{ti} + r_{it} k_{it}$ is the total income in location i
- S_{nit} is the expenditure share for agents from n to spend on goods from i

Model 2 KLR (2023): Market Clearing

- Capital market clearing means landlords' income equals payments for its use

$$r_{it} k_{it} = \frac{1-\mu}{\mu} w_{it} l_{it} \tag{9}$$

- This comes from the basic property of the CD production function

Model 2 KLR (2023): General Equilibrium

- Now we have introduced the basic environment of this model
- Let's continue to list all equations characterizing the GE
- We have state variables $\{l_{i0}, k_{i0}\}$
- And endogenous variables $\{l_{it}, k_{it}, w_{it}, R_{it}, v_{it}\}_{t=0}^{\infty}$

Model 2 KLR (2023): General Equilibrium - Capital Accumulation

- Using definition of R and capital market clearing (9), we have:

$$R_{it} = \left(1 - \delta + \frac{1-\mu}{\mu}\frac{w_{it} l_{it}}{p_{it} k_{it}}\right) \tag{10}$$

- Using (2), (4), and (9), we have local price index:

$$p_{nt} = \left[\sum_{i=1}^{N} (w_{it}(\frac{1-\mu}{\mu})^{1-\mu}(l_{it}/k_{it})^{1-\mu}\tau_{ni}/z_i)^{-\theta}\right]^{-1/\theta} \tag{11}$$

Model 2 KLR (2023): General Equilibrium - Capital Accumulation

- Therefore, the law of motion for capital is:

$$k_{it+1} = (1 - \varsigma_{it}) \left(1 - \delta + \frac{w_{it} l_{it}}{p_{it} k_{it}} \right) k_{it} \tag{12}$$

$$\varsigma_{it}^{-1} = 1 + \beta^{\psi} (E_t[R_{it+1}^{\frac{\psi-1}{\psi}} \varsigma_{it+1}^{-\frac{1}{\psi}}])^{\psi} \tag{13}$$

Model 2 KLR (2023): General Equilibrium - Goods Market Clearing

- From goods market clearing (8) and capital market clearing (9), we have:

$$w_{it} l_{it} = \sum_{n=1}^{N} S_{nit} w_{nt} l_{nt} \tag{14}$$

- We can derive expenditure share S_{nit} using (2) in a CES demand system:

$$S_{nit} = \frac{\left(w_{it}(l_{it}/k_{it})^{1-\mu}\tau_{ni}/z_i\right)^{-\theta}}{\sum_{m=1}^{N}\left(w_{mt}(l_{mt}/k_{mt})^{1-\mu}\tau_{nm}/z_m\right)^{-\theta}} \tag{15}$$

- Conversely, we can define T_{int} as the income share of exporter i to importer n:

$$T_{int} \equiv \frac{S_{nit} w_{nt} l_{nt}}{w_{it} l_{it}} \tag{16}$$

Model 2 KLR (2023): General Equilibrium - Worker Value Function

- Using basic Logit properties and plug in the indirect utility function for current period utility, we have closed form worker value function:

$$v_{nt}^{w} = \underbrace{lnb_{nt} + ln(\frac{w_{nt}}{p_{nt}})}_{\text{Current utility}} + \underbrace{\rho ln \sum_{g=1}^{N} \left(exp(\beta E_t v_{gt+1}^{w})/\kappa_{gnt}\right)^{1/\rho}}_{\text{Future value across all potential location choices}} \tag{17}$$

- where $v_{nt}^{w} \equiv E_{\epsilon}[V_{nt}^{w}]$, $E_t[v_{gt+1}^{w}] = E_t E_{\epsilon}[V_{nt+1}^{w}]$ is taken over future fundamentals
- There are two sources of uncertainty here: future preference shock, future fundamental
- Small letter v is the expectation of big letter V, after resolving the uncertainty of only future preference shock

Model 2 KLR (2023): General Equilibrium - Population Flow

- Population flow can be calculated as:

$$l_{gt+1} = \sum_{i=1}^{N} D_{igt} l_{it} \tag{18}$$

- Using basic Logit properties, we have closed form migration probability:

$$D_{igt} = \frac{(exp(\beta E_t v^w_{gt+1})/\kappa_{git})^{1/\rho}}{\sum_{m=1}^{N}(exp(\beta E_t v^w_{mt+1})/\kappa_{mit})^{1/\rho}}, \quad E_{git} \equiv \frac{l_{it} D_{igt}}{l_{gt+1}} \tag{19}$$

- D_{igt} is the share of workers from i to outmigrate to g
- E_{git} is the share of workers from i for all inmigrants in location g

Model 2 KLR (2023): General Equilibrium

- We now define the General Equilibrium of this model

Definition 1 in KLR (2023)

Given the state variables $\{l_{i0}, k_{i0}\}$ in each location in an initial period $t = 0$, an equilibrium is a stochastic process of wages, capital returns, expected values, mass of workers, and stock of capital in each location $\{w_{it}, R_{it}, v_{it}, l_{it+1}, k_{it+1}\}_{t=0}^{\infty}$ measurable with respect to the fundamental shocks up to time t $\{z_{is}, b_{is}\}_{s=1}^{t}$, and solves the value function (17), the population flow condition (18), the goods market clearing condition (14), and the capital market clearing and accumulation condition (12), with the saving rate determined by Lemma 1.

Model 2 KLR (2023): General Equilibrium

- This model can be generalized to the following components
  - Shocks to trade and migration costs
  - Agglomeration forces
  - Multiple sectors
  - Input-output linkages
  - Residential capital (Housing market)
  - Non-employment

Model 2 KLR (2023): Solving the Model

- There are three methods to solve this complicated dynamic spatial model
  - 1. Dynamic hat-algebra: One more set of equations than CDP (2019)
  - 2. Invert the model and solve unobserved fundamentals in level
  - 3. Linearize the model to have an approximated solution
- Let's consider methods 1 and 2
- Method 3 is out of our scope, you can learn it by yourself
- Of course you can also parameterize unobserved fundamental and then solve the model

Model 2 KLR (2023): Dynamic Hat-algebra

- Again we denote $\dot{x}_{it+1} = x_{it+1}/x_{it}$
- All blue terms are fundamental changes

Proposition 2 in KLR (2023)

Given an initial observed allocation of the economy, $l_{i0}, k_{i0}, k_{i1}, S_{ni0}, D_{ni-1}$, and a convergent sequence of future changes in fundamentals under perfect foresight: $\{\dot{z}_{it}, \dot{b}_{it}, \dot{\tau}_{ijt}, \dot{\kappa}_{ijt}\}$, the solution for the sequence of changes in the model's endegenous variables does not require information on the level of fundamentals: $\{z_{it}, b_{it}, \tau_{ijt}, \kappa_{ijt}\}$.

Model 2 KLR (2023): Dynamic Hat-algebra

Proof of Proposition 2 in KLR (2023): Appendix B.3

The GE in time difference can be solved using the following system of nonlinear equations:

$$\dot{D}_{igt+1} = \frac{\dot{u}_{gt+2}/(\dot{\kappa}_{git+1})^{1/\rho}}{\sum_m D_{imt}\,\dot{u}_{mt+2}/(\dot{\kappa}_{mit+1})^{1/\rho}} \tag{20}$$

$$\dot{u}_{it+1} = \left(\dot{b}_{it+1}\frac{\dot{w}_{it+1}}{\dot{p}_{it+1}}\right)^{\frac{\beta}{\rho}} \left(\sum_{g=1}^{N} D_{igt}\,\dot{u}_{gt+2}/(\dot{\kappa}_{git+1})^{\frac{1}{\rho}}\right)^{\beta} \tag{21}$$

$$\dot{p}_{it+1} = \left(\sum_{m=1}^{N} S_{imt}\left(\dot{\tau}_{imt+1}\dot{w}_{mt+1}\left(\dot{l}_{mt+1}/\dot{k}_{mt+1}\right)^{1-\mu}/\dot{z}_{mt+1}\right)^{-\theta}\right)^{-1/\theta} \tag{22}$$

$$l_{gt+1} = \sum_{i=1}^{N} D_{igt}\,l_{it} \tag{23}$$

where we define $u_{it} \equiv \exp\left(\frac{\beta}{\rho} v_{it}^{w}\right)$.

Model 2 KLR (2023): Dynamic Hat-algebra

Proof of Proposition 2 in KLR (2023): Appendix B.3

$$\dot{w}_{it+1}\dot{l}_{it+1} = \sum_{n=1}^{N} \frac{S_{nit+1} w_{nt} l_{nt}}{\sum_{k=1}^{N} S_{kit} w_{kt} l_{kt}} \dot{w}_{nt+1}\dot{l}_{nt+1} \tag{24}$$

$$\dot{S}_{nit+1} \equiv \frac{\left(\dot{\tau}_{nit+1}\dot{w}_{it+1}\left(\dot{l}_{it+1}/\dot{k}_{it+1}\right)^{1-\mu}/\dot{z}_{it+1}\right)^{-\theta}}{\sum_{k=1}^{N} S_{nkt}\left(\dot{\tau}_{nkt+1}\dot{w}_{kt+1}\left(\dot{l}_{kt+1}/\dot{k}_{kt+1}\right)^{1-\mu}/\dot{z}_{kt+1}\right)^{-\theta}} \tag{25}$$

$$\varsigma_{it+1} = \beta^{\psi} R_{it+1}^{\psi-1}\frac{\varsigma_{it}}{1-\varsigma_{it}} \tag{26}$$

$$k_{it+1} = (1-\varsigma_{it})\, R_{it} k_{it} \tag{27}$$

$$(R_{it} - (1-\delta)) = \frac{\dot{p}_{it+1}\dot{k}_{it+1}}{\dot{w}_{it+1}\dot{l}_{it+1}}\left(R_{it+1} - (1-\delta)\right) \tag{28}$$

Model 2 KLR (2023): Dynamic Hat-algebra

- This is a system very similar to the one in CDP (2019)
- Equations (20), (21), (22), (23), (24), and (25) are the same
- These equations characterize migration and trade
- Equations (26), (27), and (28) are new
- They characterize investment decisions and capital accumulation

Model 2 KLR (2023): Dynamic Hat-algebra

- With this DHA proposition at hand, new we design the algorithm to solve the transition path
- The question we ask is as follows
- We have initial allocation $l_{i0}, k_{i0}, k_{i1}, S_{ni0}, D_{ni,-1}$, on a transition path to some unknown steady-state
- Given an anticipated sequence of changes in fundamentals $\dot{z}, \dot{b}, \dot{\tau}, \dot{\kappa}$
- What is the transition path? How to solve it?

Model 2 KLR (2023): Dynamic Hat-algebra

- With Proposition 2 at hand, we can solve the model using the following contraction algorithm
  - 1. Guess a path of $\{\dot{u}\}$, where we define $u_{it} \equiv \exp\left(\frac{\beta}{\rho} v_{it}^{w}\right)$, and a path of landlord consumption rates $\{\varsigma_t\}$, both converging by period $T+1$
  - 2. Set the rental rate R_{i1} in $t=1$ according to the guessed consumption rates and the observed k_{i0}, k_{i1} using (27)
  - 3. Use (20) to derive migration share $\{D_t\}_{t=1}^{T+1}$
  - 4. Use (23) to derive labor distribution across locations
  - 5. Use $l_t, l_{t-1}, k_t, k_{t-1}, S_{t-1}$ to solve the static trade subproblem
    - (a) Solve $\dot{w}_{t+1}, S_{t+1}$ rolling forward period by period using (24) and (25)
    - (b) Solve prices using (22)
    - (c) Solve rental rates $\dot{R}_{t+1}$ using (28)
    - (d) Solve capital $\dot{k}_{t+2}$ using (27)
  - 6. Solve backwards for $\dot{u}_t$ using (21)
  - 7. Solve backwards for ς_t using (26)
  - 8. Update $\dot{u}_t, \varsigma_t$, repeat Step 2 to Step 7 until convergence

Model 2 KLR (2023): Model Inversion

- Except for DHA, we can also solve the levels of the unobserved fundamentals
- The solution is much more complicated than in the static case
- However, it is still doable for us
- The order of the solution is $\tau \to z \to \kappa \to b$

Model 2 KLR (2023): Model Inversion

- Step 1: we recover bilateral trade frictions τ_{nit} from observed trade shares S_{nit}

$$\frac{S_{nit}S_{int}}{S_{nnt}S_{iit}} = \left(\frac{\tau_{nit}\tau_{int}}{\tau_{nnt}\tau_{iit}}\right)^{-\theta} = (\tau_{nit})^{-2\theta} \tag{29}$$

- We normalize own trade cost to be 1, $\tau_{nnt} = 1$
- We assume symmetric trade cost, $\tau_{nit} = \tau_{int}$

Model 2 KLR (2023): Model Inversion

- Step 2: we recover productivity z from observed population l_{it}, wage w_{it}, capital stock k_{it}, and solved trade cost τ_{nit} using

$$w_{it} l_{it} = \sum_{n=1}^{N} \frac{\left(w_{it} \left(l_{it} / k_{it} \right)^{1-\mu} \tau_{nit} / z_{it} \right)^{-\theta}}{\sum_{m=1}^{N} \left(w_{mt} \left(l_{mt} / k_{mt} \right)^{1-\mu} \tau_{nmt} / z_{mt} \right)^{-\theta}} w_{nt} l_{nt} \tag{30}$$

Model 2 KLR (2023): Model Inversion

- Step 3: we recover bilateral migration cost κ from observed migration flows D_{igt} using

$$\frac{D_{igt} D_{git}}{D_{ggt} D_{iit}} = \left(\frac{\kappa_{git} \kappa_{igt}}{\kappa_{ggt} \kappa_{iit}} \right)^{-1/\rho} = (\kappa_{git})^{-2/\rho} \tag{31}$$

- We normalize own migration cost to be 1, $\kappa_{iit} = 1$
- We assume symmetric migration cost, $\kappa_{git} = \kappa_{igt}$

Model 2 KLR (2023): Model Inversion

- Step 4: we recover expected value of living v from observed population l_{it} and solved migration cost κ using

$$l_{gt+1} = \sum_{i=1}^{N} \frac{\left(\exp\left(\beta v_{gt+1}^{w}\right) / \kappa_{git}\right)^{1/\rho}}{\sum_{m=1}^{N} \left(\exp\left(\beta v_{mt+1}^{w}\right) / \kappa_{mit}\right)^{1/\rho}} l_{it} \tag{32}$$

Model 2 KLR (2023): Model Inversion

- Step 5: we recover amenity b from observed trade share S, migration flow D, and solved productivity z, expected value v using

$$\ln b_{it} = \left(v_{it}^{w} - v_{it+1}^{w}\right) + (1-\beta)v_{it+1}^{w} - \ln \frac{S_{iit}^{-\frac{1}{\beta}}}{(D_{iit})^{\rho}} - \ln z_{it} - (1-\mu)\ln\left(\frac{k_{it}}{l_{it}}\right) \quad (33)$$

Model 2 KLR (2023): Model Inversion

- With all these solved unobserved fundamentals, we can solve the transition path to the steady-state
- We simulate the model forward using traditional contraction algorithm

Conclusion

- Dynamic Spatial Equilibrium Model is much much harder to solve compared with the static one
- But as you can see from the lectures, they share similar modeling patterns and solving techniques
- I really hope you guys to do some work on this
- Of course, start from replicating or mimicing the model in a China topic

References

Kleinman, Benny, Ernest Liu, and Stephen J Redding. 2023. "Dynamic Spatial General Equilibrium." *Econometrica* 91 (2):385–424.